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Bumpy metric theorem in the sense of Mañé for non-convex Hamiltonian vector fields

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Théorème des métriques bosselées au sens de Mañé pour les champs de vecteurs Hamiltonien non convexe

Résumé

Une propriété est générique au sensé de Mañé si, donné un Hamiltonien $H : T^*M \to \mathbb{R}$, l'ensemble des fonctions lisses $u : M \to \mathbb{R}$ tel que $H + u$ vérifie la propriété est un sous-ensemble générique de $C^\infty(M)$. Notre objectif est de savoir dans quelle mesure la non dégénérescence de toutes les orbites périodiques dans un niveau d'énergie donné d'un Hamiltonien lisse non convexe est une propriété générique au sensé de Mañé. Où la non-dégénérescence signifie que dérivée de l'application de Poincaré ne prend pas les racines de l'unité comme une valeurs propre.
Pour atteindre cet objectif, nous obtiendrons un théorème de perturbation pour les aplication de Poincaré similaire au théorème de Rifford et Ruggiero dans le cadre convexe, et une forme normale de type Fermi sur les orbites d'un champ de vecteurs Hamiltonien non convexe. Ce sont deux outils applicables à l'étude de la dynamique des champs de vecteurs Hamiltoniens non convexes. D'autre part, nous montrerons que dans les cas convexes et non convexes, nous avons certainement besoin d'un mécanisme différent pour prouver le théorème des métrique bosselées pour les orbites symétriques. Une orbite symétrique est une orbite dont la projection sur les variétés de base comprend soit des points d'auto-intersection, soit des points à vitesse nulle. Ce fait a été négligé dans les études précédentes.
Une étude détaillée des formes normales locales sur les segments d'orbite d'un champ de vecteurs Hamiltonien est donnée. Cela inclut une forme normale pour les Hamiltoniens convexes, une forme normale pour les Hamiltoniens positivement homogènes qui implique la forme normale de Li-Nienberg pour les métriques de Finsler, et comme nous l'avons mentionné une forme normale pour les Hamiltoniens non convexes. De cette façon, nous éliminons la confusion qui existe dans la littérature entre la forme normale de Li-Nirenberg et une forme normale souhaitée similaire pour les champs de vecteurs Hamiltoniens convexes.

Mots clés : Hamiltonien non convexe, théorème des métriques bosselées, généricité au sens de Mañé

Bumpy metric theorem in the sense of Mañé for non-convex Hamiltonian vector fields

Abstract

A property (p) of smooth Hamiltonian vector fields is called Mañé-generic whenever the set of smooth potentials u such that $H + u$ satisfies the property (p) is a generic subset.
Given a non-convex smooth Hamiltonian $H : T^*M \to \mathbb{R}$ which is defined on the cotangent bundle of a smooth manifold M, our goal in this thesis is to know that to what extent non-degeneracy of all periodic orbits in a given energy level of H is a Mañé generic property. Where by a periodic non-degenerate orbit we mean a periodic orbit that its associated linearized Poincaré map does not take roots of unity as an eigenvalue.
To that end, we will achieve a perturbation theorem for linearized Poincaré maps similar to Rifford and Ruggiero's theorem in the convex setting, and a Fermi type normal form on orbits of a non-convex Hamiltonian vector field. These are two applicable tools in the study of non-convex Hamiltonian vector fields. At the other hand, we will show that in both convex and non-convex cases we certainly need a different machinery to prove the bumpy metric theorem for symmetric orbits. A symmetric orbit is an orbit that its projection on the base manifolds includes either self-intersection points or points with zero velocity. This fact was overlooked in previous studies.
A detailed study of local normal forms on orbit segments of a Hamiltonian vector field is given. That includes a normal form for convex Hamiltonians, a normal form for positively homogeneous Hamiltonians that implies Li-Nienberg normal form for Finsler metrics, and as we mentioned a normal form for non-convex Hamiltonians. In this way, we remove the confusion that exists in the literature between Li-Nirenberg normal form and a similar desired normal form for convex Hamiltonian vector fields.

Keywords: non-convex Hamiltonians, bumpy metric theorem, Mañé-generic properties

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Introduction of the Thesis

A closed geodesic of a Riemannian metric is called *non-degenerate* if its associated linearized Poincaré map does not have a root of unity as an eigenvalue. A Riemannian metric is called *bumpy* whenever all its closed geodesics are non-degenerate. We are able to use a similar notion of non-degeneracy for periodic orbits of a Hamiltonian vector field. Consider $R^r(M)$ as the set of all C^r Riemannian metrics on a smooth manifold M. We equip $R^r(M)$ with the Whitney C^r topology. Let $B^r(M) \subset R^r(M)$ be the set of all C^r bumpy metrics. The following theorem is known as the *bumpy metric theorem*.

Theorem A. *Assume that M is a given smooth manifold. For $r \geq 2$, $B^r(M)$ is a G_δ dense subset of $R^r(M)$.*

In other words, the bumpy metric theorem states that the property (p) explained as follows

(p) all closed geodesics are non-degenerate

is a C^r *generic property* of $R^r(M)$ i.e. the set of all metrics that satisfy property (p) is a G_δ dense subset of $R^r(M)$. Note that a G_δ subset is a countable union of open subsets.
The bumpy metric theorem is stated by Abraham [Abr70], and a complete proof of it is achieved after contributions of Abraham [AR67; Abr70], Klingenberg and Takens [KT72], and Anosov [Ano83].

Consider $\mathcal{H}^r(T^*M)$ as the set of all C^r Hamiltonians defined on the cotangent bundle of a smooth manifold M. Assume that $\mathcal{H}^r(T^*M)$ is endowed with Whitney C^r topology. An energy level $H^{-1}(k)$ of a Hamiltonian $H : T^*M \to \mathbb{R}$ is *regular* whenever the Hamiltonian vector field of H does not vanish on $H^{-1}(k)$. Concerned to $\mathcal{H}^r(T^*M)$ where $r \geq 2$, Robinson [Rob70a; Rob70b] obtained a similar result as the bumpy metric theorem:

Theorem B. *Assume that M is a smooth manifold and $r \geq 2$. Let $k \in \mathbb{R}$ be given. If we define $\mathcal{K}_k^r(T^*M)$ as the set of Hamiltonians $H \in \mathcal{H}^r(T^*M)$ such that $H^{-1}(k)$ is a regular energy level and all periodic orbits in the k-energy level of H are non-degenerate, then $\mathcal{K}_k^r(T^*M)$ is a G_δ dense subset of $\mathcal{H}^r(T^*M)$.*

In this thesis, we study generic properties of $\mathcal{H}^\infty(T^*M)$ with respect to a much more restrictive concept of genericity compared to genericity in terms of the Whitney topologies.
The idea of perturbing a given Hamiltonian $H : T^*M \to \mathbb{R}$ by adding a *potential* $u \in C^\infty(M)$ is suggested by Ricardo Mañé [Mn96]. Where by a potential we mean a function that only depends on the based manifold. To put it another way, $H + u$ where $u \in C^\infty(M)$, is a *Mañé perturbation* of a smooth Hamiltonian $H : T^*M \to \mathbb{R}$.
Using the Legendre transformation, to a given smooth Riemannian metric g we can correspond a quadratic Hamiltonian $H_g(q,p) = \langle p, p \rangle_q$, where $\langle ., . \rangle_q$ is the inner product on T_q^*M induced by the Riemannian metric g. Geodesics of the metric g are the canonical projections of orbits of the Hamiltonian vector fields of H_g in a given non-zero energy level.
A smooth conformal perturbation of a Riemannian metric g is a perturbation of the form $e^{b(q)}g$

where $b(q) \in C^\infty(M)$. *Maupertuis' principle* (see [LR98], Theorem 1) implies that a C^∞-small conformal perturbation of a Riemannian metric is equivalent to perturbing its associated Hamiltonian via adding a C^∞-small potential.

A property (p) is called Mañé generic for $\mathcal{H}^\infty(T^*M)$ whenever for a given $H \in \mathcal{H}^\infty(T^*M)$ the set $\mathcal{G} = \big\{u \in C^\infty(M) \mid H+u \text{ satisfies } (p)\big\}$ is a G_δ dense subset of $C^\infty(M)$, where the topology that we have considered on $C^\infty(M)$ is the Whitney C^∞ topology.
The following theorem is known as *the bumpy metric theorem in the sense of Mañé for convex Hamiltonians*. A Hamiltonian $H(q,p) : T^*M \to \mathbb{R}$ is *convex* whenever its fiberwised Hessian i.e. $\partial^2_{p^2} H(q,p)$ is positive-definite for all $(q,p) \in T^*M$.

Theorem C. *Suppose that M is a given smooth manifold with dimension $d+1$, where $d \geq 1$, and $H : T^*M \to \mathbb{R}$ is a convex smooth Hamiltonian. For a given $k \in \mathbb{R}$, there exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for all $u \in \mathcal{G}$, $(H+u)^{-1}(k)$ is a regular energy level and all closed orbits in $(H+u)^{-1}(k)$ are non-degenerate.*

Oliveira [Oli08] studied the case $d = 1$ of Theorem C above. A so-called *perturbation theorem* —that is obtained later by Rifford and Ruggiero [RR12]— was missing to extend the outcome of Oliveira's studies to higher dimensions.

Definition 1. *Assume that $\theta(t) = \big(Q(t), P(t)\big)$ is a periodic orbit of Hamiltonian vector field of $H : T^*M \to \mathbb{R}$. Moreover, suppose that T is the minimum period of $\theta(t)$. A time t_0 is called neat for $\theta(t)$ if $\dot{Q}(t_0) \neq 0$ and $Q(t_0)$ is not a self-intersection of $Q(t)$ i.e. there does not exist a time $s \neq t_0$ modulo T such that $Q(s) = Q(t_0)$. If t_0 is a neat time for $\theta(t)$, then we call $\theta(t_0)$ a neat point.*

Both [RR12] and [Oli08] are implicitly assuming that, given a convex Hamiltonian H, all closed orbits θ of Hamiltonian vector field of H are admitting a neat time. This assumption is not true. For a Hamiltonian of the form $H(q,p) = g(p,p) + u(q)$, where g is a Riemannian metric and u is a potential, Kozlov [Koz76] proved the existence of *periodic librations*, where the term *libration* refers to an orbit without any neat time. An orbit of a Hamiltonian vector field with no neat time is also called a *symmetric orbit*. Look at Section 1 of [Dev76] for example where features of symmetric orbits of reversible mechanical systems are studied.
We formulate the main assertion of [RR12] (Theorem 1.2 of [RR12]) as Theorem D below, making explicit the unstated assumption.

Definition 2. *Assume that $H : T^*M \to \mathbb{R}$ is a smooth Hamiltonian which is defined on the cotangent bundle of a smooth manifold M. Suppose $\theta(t)$ is an orbit of the Hamiltonian vector field of H. We use the notation $C^\infty_\theta(M)$ for the set of admissible potentials concerned to θ which is defined as follows*

$$C^\infty_\theta(M) := \big\{u \in C^\infty(M) \mid u\big(\pi \circ \theta(t)\big) = 0,\ du\big(\pi \circ \theta(t)\big) = 0,\ \textit{for all } t \in \mathbb{R}\big\}.$$

A perturbation alike $H+u$ is an admissible perturbation with respect to θ whenever $u \in C^\infty_\theta(M)$.

In the following theorem, $Sp(2d)$ refers to the set of all symplectic matrices of dimension $2d \times 2d$, and by a *regular periodic orbit* we mean a periodic orbit that is not a stationary point.

Theorem D (Perturbation theorem for convex Hamiltonians)**.** *Suppose that $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth convex Hamiltonian and $\theta(t) \in H^{-1}(k)$ is a regular periodic orbit of Hamiltonian vector field of H. Moreover, assume that $\theta(t)$ admits a neat time. Consider*

$$P_u(\theta, \Sigma) : \Sigma \cap (H+u)^{-1}(k) \to \Sigma \cap (H+u)^{-1}(k), \quad u \in C^\infty_\theta(\mathbb{R}^{d+1}),$$

as the restricted Poincaré map with respect to θ *and Hamiltonian vector field of* $H+u$*, where* Σ *is a transverse section to* $\theta(t)$*. The map* $F(\theta, H+u)$ *defined as*

$$C^\infty_\theta(\mathbb{R}^{d+1}) \ni u \mapsto dP_u \in Sp(2d)$$

is weakly open.

A mapping is *weakly open*, if the image of every non-empty open set has a non-empty interior. For a given periodic orbit $\theta(t)$ of a Hamiltonian vector field, by the *restricted Poincaré map* in the above statement we mean the restriction of the Poincaré map to the energy level that includes $\theta(t)$.

Once more, consider a Hamiltonian of the form $H(q,p) = g(p,p) + u(q)$ where g is a Riemannian metric and u is a potential. In Section 2.3 we will demonstrate that Theorem D does not hold for a periodic symmetric orbit of such a Hamiltonian H. That means "$\theta(t)$ admits a neat time" is a necessary assumption for Theorem D. In consequence, contrary to what is believed in the literature, Theorem C is still open. In this thesis, we do not attempt to prove Theorem C for the case of periodic librations.

The proof given in [RR12] of Theorem D is founded on Lemma C1 of [FR15]; A local normal form on orbit segments that is similar to *Fermi coordinates* for Riemannian metrics. See [Kli78], [GMK68], and Section 5 of [Con10] for Fermi coordinates.
Lemma C1 of [FR15] asserts that a Fermi type symplectic coordinates for a Hamiltonian $H(q,p) : T^*M \to \mathbb{R}$ is achievable by performing only *homogeneous fibered symplectic change of coordinates* i.e. change of coordinates of the form $\Psi(q,p) = \big(\varphi(q), [d\varphi^{-1}(q)]^T p\big)$ where φ is a diffeomorphism. A symplectomorphism is called *fibered* whenever it preserves the vertical fibrations. A fibered symplectomorphism is *homogeneous* if it preserves the zero section, and it is *vertical* if it is of the form $\Psi(q,p) = \big(q, p + dg(q)\big)$ where g is a real valued C^2 function. In Chapter 1, we prove that Lemma C1 of [FR15] is wrong. Furthermore, we obtain an alternative normal form that is weaker than Lemma C1 of [FR15] but it is sufficient to save the proof of Theorem D which is given in [RR12].

Besides homogeneous fibered symplectic change of coordinates, the alternative normal form allows to perform *vertical fibered symplectomorphisms*, and *conformal reparametrizations* i.e. multiplying a function that only depends on the position variable to the Hamiltonian.

Theorem 1 (Alternative normal form)**.** *Assume that* $\underline{H}(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ *is a given convex smooth Hamiltonian. Consider* $\underline{\theta}(t) = \big(\underline{Q}(t), \underline{P}(t)\big)$ *as a given orbit of the Hamiltonian vector field of* $\underline{H}$ *such that* $\dot{\underline{Q}}(0) \neq 0$ *and* $\underline{H}(\underline{\theta}) = k$*. There exist a smooth fibered symplectomorphism* $\Psi(q,p) : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$*, a positive real number* δ*, and a smooth function* $z(q) : \mathbb{R}^{d+1} \to \mathbb{R}$ *such that* $\big(Q(t), P(t)\big) := \Psi^{-1}(\underline{\theta})$ *is an orbit of the Hamiltonian vector field of*

$$H(q,p) := z(q)\big(\underline{H} \circ \Psi(q,p) - k\big).$$

Moreover, for all $t \in [-\delta, \delta]$*, we have*

(1) $Q(t) = te_1, \quad e_1 = (1, 0_d)$
(2) $P(t) = 0$
(3) $\partial^2_{p_1\hat{p}} H(te_1, 0) = 0$
(4) $\partial^2_{qp} H(te_1, 0) = 0$
(5) $\partial^2_{\hat{p}^2} H(te_1, 0) = I.$

In the above theorem we are using the notation $q = (q_1, \hat{q}) \in \mathbb{R} \times \mathbb{R}^d$, and $p = (p_1, \hat{p}) \in \mathbb{R} \times \mathbb{R}^d$.

Li and Nirenberg [LN05] have obtained a local normal form for Finsler metrics (see [LN05], Lemma 3.1) that is similar to Fermi coordinates. Lemma C1 of [FR15] employs Li-Nirenberg normal form to acquire a similar normal form for convex Hamiltonians which are not necessarily homogeneous with respect to momentum variable. In Section 1.3, using our original methods, we prove a normal form for positively homogeneous Hamiltonians; See Theorem 1.3.1 which implies the Li-Nirenberg normal form. The proof of this normal form helps to clarify the differences between the Li-Nirenberg normal form and a similar desired normal form for non-homogeneous Hamiltonians.

Using geometric control methods, Rifford and Ruggiero [RR12] are not only giving a bright perspective of the perturbation theorem, but also an essential contribution to Theorem C. The normal form given in Theorem 1 aids to reduce Theorem D to a control problem.
Consider $\phi^t(x,u)$ as the Hamiltonian flow of $H+u$, and $\theta(t)$ as a given regular periodic orbit of H. In the coordinates given by Theorem 1, let $R^t_u : \Lambda_0 \cap (H+u)^{-1}(0) \to \Lambda_t \cap (H+u)^{-1}(0)$, be the perturbed *restricted transition map* where Λ_t is defined as $\Lambda_t := \{q_1 = t\}$, and $u \in C^\infty_\theta(\mathbb{R}^{d+1})$. Using the definition of the Hamiltonian flow, we have the equation

$$\frac{d}{dt}\partial_x\phi^t(0,u) = \mathbb{J}\partial^2_{x^2}(H+u)(te_1,0)\partial_x\phi^t(0,u), \quad \mathbb{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{1}$$

which is known as the *Jacobi equation* in some contexts. Equation (1) can be seen as a control problem where $-\partial^2_{q^2}u(te_1)$ is the control. In the coordinate system introduced by Theorem 1, we can view equation (1) as two uncoupled equations. In Section 2.2.1, we will show that the solution of one of these uncoupled equations is identical to the differential of the perturbed transition map at 0, namely $dR^t_u(0)$. In Riemannian dynamics, a similar application of Fermi coordinates is discussed in Section 5 of [Con10].

One of our goals in this thesis is to know that to what extent Theorem D holds after removing the assumption of convexity. The following definition has a central role in our study of Hamiltonians that are not necessarily convex.

Definition 3. *A smooth Hamiltonian $H : T^*M \to \mathbb{R}$ is fiberwised iso-energetically non-degenerate at a point $(q,p) \in T^*M$ if*

$$\det \begin{bmatrix} \partial^2_{p^2}H(q,p) & \partial_pH(q,p) \\ [\partial_pH(q,p)]^T & 0 \end{bmatrix} \neq 0.$$

In other words, Hamiltonian $H : T^*M \to \mathbb{R}$ is fiberwise iso-energetically non-degenerate at a point $(q,p) \in T^*M$ whenever p is a regular point of the function $H(q,.) : T^*_qM \to \mathbb{R}$, and the Hessian of $H(q,.)$ is non-degenerate on the kernel of its differential $ker\big(dH(q,.)\big)$. For a given smooth Hamiltonian $H : T^*M \to \mathbb{R}$, define

$$\mathit{\Gamma}_H := \left\{(q,p) \in T^*M \mid \quad \det \begin{bmatrix} \partial^2_{p^2}H(q,p) & \partial_pH(q,p) \\ [\partial_pH(q,p)]^T & 0 \end{bmatrix} = 0\right\}. \tag{2}$$

Note that H is iso-energetically non-degenerate at (q,p) if and only if $(q,p) \notin \mathit{\Gamma}_H$.
For a convex Hamiltonian $H : T^*M \to \mathbb{R}$, the set $\mathit{\Gamma}_H$ is all the points $(q,p) \in T^*M$ such that $\partial_pH(q,p) = 0$. Therefore, whenever a smooth Hamiltonian H is convex, exactly one point per fiber belongs to $\mathit{\Gamma}_H$.

The purpose of Chapter 2 is to prove Theorem 3 below which we call the perturbation theorem for non-convex Hamiltonians.

Theorem 3. *Suppose that $\theta(t) \in H^{-1}(k)$ is a regular periodic orbit of a smooth Hamiltonian $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$. We consider*

$$P_u(\theta, \Sigma) : \Sigma \cap (H+u)^{-1}(k) \to \Sigma \cap (H+u)^{-1}(k), \quad u \in C^\infty_\theta(\mathbb{R}^{d+1}),$$

as the restricted Poincaré map with respect to $\theta(t)$ and Hamiltonian vector field of $H+u$, where Σ is a transverse section to $\theta(t)$. Assume that $\theta(t)$ admits a neat time t_0 such that $\theta(t_0) \notin \Gamma_H$. Then, the map $F(\theta, H+u)$ defined as

$$C^\infty_\theta(\mathbb{R}^{d+1}) \ni u \mapsto dP_u \in Sp(2d)$$

is weakly open.

We prove Theorem 3 using a generalization of the alternative normal form for non-convex Hamiltonians. See Theorem 1.4.1. Our proof of Theorem 3 relies on similar geometric control methods as Rifford and Ruggiero [RR12] have applied.
Note that for a convex Hamiltonian H, if t_0 is a neat time for an orbit $\theta(t)$ of the Hamiltonian vector field of H, then the condition $\theta(t_0) \notin \Gamma_H$ is automatically satisfied. Therefore, Theorem D is a particular case of Theorem 3.
In Section 2.3, we demonstrate that the assumption "$\theta(t)$ admits a neat time t_0 such that $\theta(t_0) \notin \Gamma_H$" is a necessary assumption for Theorem 3.

Before we introduce the genericity results of this thesis we wish to give a brief chronological review of related studies about generic properties of dynamical systems.
Kupka [Kup64] and Smale [Sma63] have studied typical properties of the set of C^r vector fields defined on a compact manifold M, namely $\mathcal{X}^r(M)$. They independently proved that hyperbolicity of all closed orbits, and transversality of all heteroclinic intersections are C^r generic properties for $\mathcal{X}^r(M)$ where $r \geq 1$. Peixoto [Pei67] reproves the same result —which is known as *Kupka-Smale* theorem— with different methods and generalized it for $\mathcal{X}^r(M)$ where M is a non-compact manifold. Abraham [AR67] gives a new proof of the Kupka-Smale theorem after obtaining the parametric transversality theorem. It is important to note that hyperbolicity of all closed orbits in a given energy level is not a generic property of Hamiltonian vector fields; See [MP70; Rob70a; Rob70b] for more details.
As we saw earlier, parallel studies in the context of Mañé genericity are much more recent. Inspired by [KT72], Carballo and Miranda [CM13] shows that for a given closed orbit θ of a convex Hamiltonian H, the ℓ-jets of the map $F(\theta, H+u)$ —that we have defined in Theorem D— are weakly open for all $\ell \geq 1$. The proof given in [CM13] depends on conclusions of [RR12]. Lazrag, Rifford and Ruggiero [LRR16; Laz14] proves that the restriction of the same map $F(\theta, H+u)$ to C^2 potentials is an open mapping.

In Chapter 3, we prove two genericity results, Theorem 4 and Theorem 5 below. These theorems are immediately concluding Theorem 6 which we can consider as a bumpy metric theorem in the sense of Mañé for non-convex Hamiltonians.

Theorem 4. *Assume that $H : T^*M \to \mathbb{R}$ is a smooth Hamiltonian defined on the cotangent bundle of a closed smooth manifold M. Let $\Upsilon \subset Sp(2d)$ be a given F_σ nowhere dense subset invariant under congugacy. For a given $k \in \mathbb{R}$, there exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for all $u \in \mathcal{G}$ the k-energy level of $H+u$ is regular; Moreover, if $\theta(t) \in (H+u)^{-1}(k)$ is a periodic orbit that admits a neat time t_0 such that $\theta(t_0) \notin \Gamma_H$, then the linearized restricted Poincaré map associated to θ and Hamiltonian vector field of $H+u$ does not belong to Υ.*

Note that a subset is called F_σ if it is the complement of a G_δ subset.

The proofs given by Oliveira [Oli08] and Anosov [Ano83] of bumpy metric theorems are

including three main phases: Besides a machinery to perturb the linearized Poincaré map associated to a given periodic orbit (Theorem 4.5 in [Oli08] and Theorem 2 in [KT72]), both proofs in [Ano83] and [Oli08] are applying the *parametric transversality theorem* which is a generalization of Thom's transversality theorem. See Theorem 19.1 of [AR67]. Furthermore, Anosov and Oliveira are applying an induction on recurrence of periods of orbits; Look at the proof of Lemma 3.6 in [Oli08] for example.
The proof that we give for Theorem 4 in Section 3.2.1 has a crucial dependence on the perturbation theorem for non-convex Hamiltonians. However, we do not use induction on recurrence of periods, and we prove a variant of parametric transversality theorem tacitly during the proof of Theorem 4. In this way, the proof of Theorem 4 would be comprehensible after recalling a few standard facts in general topology. Nevertheless, we face a new challenge to prove the bumpy metric theorem in the non-convex setting. That is to investigate, given a smooth Hamiltonian H, to what extent all non-symmetric orbits of $H + u$ are admitting a neat time in the complement of $\varGamma_H$ where u is a generic smooth potential. See Theorem 5 below in which we refer to the following hypothesis.

Hypothesis 1. *The subset $\varGamma_H \subset T^*M$ is contained in a countable union of submanifolds of positive codimension which are transversal to the vertical fibrations.*

Theorem 5. *Let $H : T^*M \to \mathbb{R}$ be a smooth Hamiltonian defined on the cotangent bundle of a smooth manifold M. Assume that $\varGamma_H$ satisfies Hypothesis 1. There exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for all $u \in \mathcal{G}$, the Hamiltonian vector field associated to $H + u$ has the following property:*

For each orbit $\theta(t)$ of $H + u$ and each time t_0 such that $\partial_p H\big(\theta(t_0)\big) \neq 0$, there exist an open neighborhood $I \subset \mathbb{R}$ around t_0 so that $\theta(I \setminus t_0) \cap \varGamma_H = \emptyset$.

If t_0 is a neat time of an orbit $\theta(t)$ of Hamiltonian vector field of H, then there exists an open segment of $\theta(t)$ such that it includes t_0 and it consists of neat points only. That is to say admitting a neat time is an open condition for an orbit $\theta(t)$. Hence, Theorem 5 instantly implies that the property (g) described below is a Mañé-generic property for $\hat{\mathcal{H}}^\infty(T^*M)$ which denotes for all Hamiltonians $H \in \mathcal{H}^\infty(T^*M)$ such that $\varGamma_H$ satisfies Hypothesis 1.

(g) all orbits that are admitting a neat time are also admitting a neat time in the complement of $\varGamma_H \subset T^*M$.

In other words, for a given Hamiltonian $H \in \hat{\mathcal{H}}^\infty(T^*M)$, the set of potentials $\{u \in C^\infty(M) \mid H + u \text{ satisfies } (g)\}$ is a G_δ dense subset of $C^\infty(M)$. Therefore, Theorem 4 and Theorem 5 imply the following theorem.

Theorem 6. *Assume that $H : T^*M \to \mathbb{R}$ is a smooth Hamiltonian defined on the cotangent bundle of a smooth manifold M. Suppose that $\varGamma_H$ satisfies Hypothesis 1. Let $\Upsilon \subset Sp(2d)$ be a F_σ nowhere dense subset invariant under congugacy. For a given $k \in \mathbb{R}$, there exists a G_δ dense subset $\mathcal{G} \subset C^\infty(M)$ such that for all $u \in \mathcal{G}$ the Hamiltonian $H + u$ has the following property:*

$(H + u)^{-1}(k)$ is a regular energy level. Furthermore, if $\theta(t) \in (H + u)^{-1}(k)$ is a periodic orbit that admits a neat time, then the linearized restricted Poincaré map associated to $\theta(t)$ and Hamiltonian vector field of $H + u$ does not belong to Υ.

If we choose $\Upsilon \subset Sp(2d)$ as the subset of all matrices that are taking a root of unity as their eigenvalues, then the above theorem for such subset Υ is what we are able to consider as a bumpy metric theorem in the sense of Mañé for non-convex Hamiltonians.

The results of this thesis is already published in research articles [AB21] and [AB22]. From time to time, the notations and presentation of the statements and proofs in this thesis are slightly different compared to the articles.

Chapter 1

Local normal form on orbits of a Hamiltonian vector field

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Li and Nirenberg [LN05] achieved a local normal form on geodesics of Finsler metrics. See Lemma 3.1 in [LN05] which is equivalent to Corollary 1.3.2 below. As we mentioned in the introduction of this thesis, Lemma C.1 of [FR15] attempts to apply Li-Nirenberg normal form to obtain a similar normal form for convex Hamiltonians. We represent Lemma C.1 of [FR15] as Proposition 1.2.6 below.
In Section 1.2, we prove that the statement of Lemma C.1 in [FR15] is wrong. Furthermore, we introduce an alternative normal form weaker than Lemma C.1 in [FR15] but applicable to save the proof of Theorem D given in [RR12] which is based on the wrong normal form.
In Section 1.3, Theorem 1.3.1 gives a normal form for homogeneous Hamiltonians. We prove Theorem 1.3.1 using the methods that we develop during the chapter and we show that it implies Li-Nirenberg normal form.
We devote Section 1.4 to obtain an extension of the alternative normal form for non-convex Hamiltonians. See Theorem 1.4.1 which we apply to prove the perturbation theorem for non-convex Hamiltonians in the next chapter.

In the first two chapters of this thesis, since we are studying local properties of Hamiltonian vector fields, without loss of generality we state all the results for Hamiltonians that are defined on cotangent bundle of an Euclidean space instead of cotangent bundle of a smooth manifold.

1.1 Preliminaries

1.1.1 Fibered symplectomorphisms

Definition 1.1.1. *A symplectomorphism* $\Psi : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$ *is fibered if it preserves the vertical fibrations. In other words,* Ψ *is fibered if and only if* $\Psi(q,p) = \big(\varphi(q), G(q,p)\big)$ *where* $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ *is a diffeomorphism.*

In the context of this thesis, it is necessary to restrict ourselves to perform only fibered symplectic changes of coordinates. Otherwise, a given function that depends only on the variable q would depend on the variable p after performing a symplectic change of coordinates, and that means the set of potentials would depend on coordinate system.

Definition 1.1.2. *A fibered symplectomorphism* $\Psi(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ *is homogeneous if it preserves the zero section. In other words,* Ψ *is homogeneous if and only if*

$$\Psi(q,p) = \big(\varphi(q), [d\varphi^{-1}(q)]^T p\big),$$

where $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ *is a diffeomorphism.*
A fibered symplectomorphism Ψ *is vertical if* $\Psi(q,p) = \big(q, p + dg(q)\big)$ *for a* C^2 *function* $g : \mathbb{R}^{d+1} \to \mathbb{R}$.

The following lemma approves that each fibered symplectomorphism is either homogeneous, vertical, or a composition of a homogeneous (vertical) symplectomorphism with a vertical (homogeneous) symplectomorphism.

Lemma 1.1.3. *Given a diffeomorphism* $\varphi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$*, a function* $\Psi : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$ *defined as* $\Psi(q,p) := \big(\varphi(q), G(q,p)\big)$ *is a symplectomorphism if and only if* G *satisfies*

$$G(q,p) = [d\varphi^{-1}(q)]^T p + dg(q),$$

for a C^2 *function* $g : \mathbb{R}^{d+1} \to \mathbb{R}$.

Proof. By definition of a symplectomorphism, Ψ satisfies the following

$$d\Psi\, \mathbb{J}\, [d\Psi]^T = \mathbb{J}, \quad \mathbb{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{1.1.1}$$

We expand the left side of (1.1.1) to have

$$\begin{bmatrix} d\varphi & 0 \\ \partial_q G & \partial_p G \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} [d\varphi]^T & [\partial_q G]^T \\ 0 & [\partial_q G]^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} o & d\varphi[\partial_q G]^T \\ -\partial_p G[d\varphi]^T & -\partial_p G \partial_q G + \partial_p G[\partial_q G]^T \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix};$$

Therefore,

$$d\varphi[\partial_p G]^T = I, \tag{1.1.2}$$
$$-(\partial_p G)\partial_q G + \partial_p G[\partial_q G]^T = 0. \tag{1.1.3}$$

From (1.1.2), we have $\partial_p G(q,p) = [d\varphi^{-1}(q)]^T$ which implies

$$G(q,p) = \big[d\varphi^{-1}(q)\big]^T p + f(q), \tag{1.1.4}$$

where $f : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ is a C^1 function. After differentiating equation (1.1.4) with respect to q and p variables we conclude the following equations

$$\partial_p G(q,0) = [d\varphi^{-1}(q)]^T, \tag{1.1.5}$$
$$\partial_q G(q,0) = df(q). \tag{1.1.6}$$

Now we rewrite (1.1.3) on the points $(q, 0)$

$$-\partial_p G(q,0)\partial_q G(q,0) + \partial_p G(q,0)[\partial_q G(q,0)]^T = 0. \tag{1.1.7}$$

Replacing equations (1.1.5) and (1.1.6) into (1.1.7) yields

$$[d\varphi^{-1}(q)]^T[df(q)]^T = [d\varphi^{-1}(q)]^T df(q) \Rightarrow [df(q)]^T = df(q).$$

Hence, since $[df(q)]^T = df(q)$, by Poincaré lemma a C^2 function $g : \mathbb{R}^{d+1} \to \mathbb{R}$ exists such that $f = dg$ which allows us to rewrite (1.1.4) as $G(q,p) = [d\varphi^{-1}(q)]^T p + dg(q)$. □

1.1.2 Homogeneous Lagrangian and Hamiltonians, Finsler metrics

For a convex Hamiltonian $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$, recall the Legendre-Fenchel duality which corresponds to H the Lagrangian $L : T\mathbb{R}^{d+1} \to \mathbb{R}$ defined as

$$L(q,v) = \sup_{p\in T_q^*\mathbb{R}^{d+1}} \big\{\langle p, v\rangle - H(q,p)\big\}. \tag{1.1.8}$$

Definition 1.1.4. *Consider a Lagrangian $L(q,v) : T\mathbb{R}^{d+1} \to \mathbb{R}^+$ which admits only positive values. For a fixed $\beta \in \mathbb{N}\cup\{0\}$, L is called (positively) β-homogeneous or (positively) homogeneous of degree β if for each ($r \in \mathbb{R}^+$) $r \in \mathbb{R} \setminus \{0\}$,*

$$L(q, rv) = r^\beta L(q,v).$$

L is called a Finsler metric if it satisfies the following three properties
(1) L is smooth on $v \neq 0$
(2) L is positively 1-homogeneous
(3) L is even with respect to the v variable i.e $L(q,-v) = L(q,v)$
(4) L^2 is convex i.e. $\partial^2_{v^2} L^2(q,v)$ is positive-definite for all $(q,v) \in \mathbb{R}^{d+1}$.
Similarly, we say $H(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}^+$ is (positively) β-homogeneous if

$$H(q, rp) = r^\beta H(q,p). \tag{1.1.9}$$

Lemma 1.1.8 below determines the relation between orders of homogeneity of a convex Hamiltonian and its corresponding Lagrangian. To prove this lemma, we use Euler's theorem for homogeneous functions. Further in this chapter, we apply Euler's theorem in Section 1.3 where we prove a normal form for homogeneous Hamiltonians.

Theorem 1.1.5 (Euler's theorem for homogeneous functions)**.** *Lagrangian $L(q,v) : T\mathbb{R}^{d+1} \to \mathbb{R}^+$ is β-homogeneous if and only if*

$$\big\langle v, \partial_v L(q,v)\big\rangle = \beta L(q,v). \tag{1.1.10}$$

Similarly, a Hamiltonian $H(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}^+$ is β-homogeneous if and only if

$$\big\langle p, \partial_p H(q,p) \big\rangle = \beta H(q,p). \tag{1.1.11}$$

Remark 1.1.6. *If $L(q,v) : T\mathbb{R}^{d+1} \to \mathbb{R}^+$ is a Finsler metric then it satisfies*

$$\big\langle v, \partial_v L(q,v) \big\rangle = L(q,v). \tag{1.1.12}$$

However, a Lagrangian that satisfies (1.1.12) is not necessarily even with respect to the variable v.

Corollary 1.1.7. *If $L(q,v)$ be a positively β-homogeneous Lagrangian, then $\partial_v L(q,v)$ is positively homogeneous of degree $\beta - 1$ with respect to v-variable.*
For a homogeneous Hamiltonian $H(q,p)$ of degree β, $\partial_p H(q,p)$ is positively $(\beta-1)$-homogeneous with respect to p-variable.

Proof. Differentiation (1.1.11) with respect to p yields $\big\langle p, \partial^2_{p^2} H(q,p) \big\rangle + \partial_p H(q,p) = \beta \partial_p H(q,p)$, so we have

$$\big\langle p, \partial^2_{p^2} H(q,p) \big\rangle = (\beta - 1)\partial_p H(q,p). \tag{1.1.13}$$

From equation (1.1.13) above and Euler's theorem we conclude that $\partial_p H(q,p)$ is positively homogeneous of order $\beta - 1$. □

Lemma 1.1.8. *Assume that Lagrangian $L(q,v) : T\mathbb{R}^{d+1} \to \mathbb{R}^+$ is smooth, convex, and positively β_1-homogeneous where $\beta_1 > 1$. If $H(q,p)$ is the dual Hamiltonian of L, then H is positively β_2-homogeneous where $\frac{1}{\beta_1} + \frac{1}{\beta_2} = 1$.*

Proof. By Corollary 1.1.7, $p = \partial_v L(q,v)$ is positively $(\beta_1 - 1)$-homogeneous with respect to v-variable. For a given $r > 0$ the Legendre-Fenchel duality gives

$$\begin{aligned} L(q,rv) = r^{\beta_1} L(q,v) &= \sup_{p \in T^*_q \mathbb{R}^{d+1}} \big\{ \big\langle r^{\beta_1 - 1} p, rv \big\rangle - H(q, r^{\beta_1 - 1} p) \big\} \\ &= \sup_{p \in T^*_q \mathbb{R}^{d+1}} \big\{ r^{\beta_1} \langle p, v \rangle - r^{(\beta_1 - 1)\beta_2} H(q,p) \big\}. \end{aligned} \tag{1.1.14}$$

After multiplying $r^{-\beta_1}$ to (1.1.14) we have

$$L(q,v) = \sup_{p \in T^*_q \mathbb{R}^{d+1}} \big\{ \langle p, v \rangle - r^{(\beta_1 - 1)\beta_2 - \beta_1} H(q,p) \big\}.$$

We have assumed that H is the corresponding Hamiltonian of L; Therefore, from the above equation and Legendre-Fenchel duality we conclude that $(\beta_1 - 1)\beta_2 - \beta_1 = 0$ which implies that $\frac{1}{\beta_1} + \frac{1}{\beta_2} = 1$. □

1.2 Local normal form on orbits of a convex Hamiltonian

In this section, first we introduce the alternative normal form, and afterwards we prove that the statement of Lemma C1 in [FR15] is not correct.

1.2.1 The alternative normal form

In the statement of the following theorem we are using the notation $q = (q_1, \hat{q}) \in \mathbb{R} \times \mathbb{R}^d$ and $p = (p_1, \hat{p}) \in \mathbb{R} \times \mathbb{R}^d$.

Theorem 1.2.1 (Alternative normal form). *Assume that $\underline{H}(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ is a given convex smooth Hamiltonian. Consider $\underline{\theta}(t) = \big(\underline{Q}(t), \underline{P}(t)\big)$ as a given orbit of the Hamiltonian vector field of $\underline{H}$ such that $\dot{\underline{Q}}(0) \neq 0$ and $\underline{H}(\underline{\theta}) = k$. There exist a smooth fibered symplectomorphism $\Psi(q,p) : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$, a positive real number δ, and a smooth function $z(q) : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $\big(Q(t), P(t)\big) := \Psi^{-1}(\underline{\theta})$ is an orbit of the Hamiltonian vector field of $H(q,p) := z(q)\big(\underline{H} \circ \Psi(q,p) - k\big)$. Moreover, for all $t \in [-\delta, \delta]$, we have*

(1) $Q(t) = te_1, \quad e_1 = (1, 0_d)$

(2) $P(t) = 0$

(3) $\partial^2_{p_1\hat{p}} H(te_1, 0) = 0$

(4) $\partial^2_{qp} H(te_1, 0) = 0$

(5) $\partial^2_{\hat{p}^2} H(te_1, 0) = I.$

Remark 1.2.2. *Assertions (1) and (2) imply*

(6) $\partial^2_{q_1 q} H(te_1, 0) = 0, \quad$ *for all* $\quad t \in [-\delta, \delta].$

Proof of the remark. From (1) and (2) we have $\partial_q H(te_1, 0) = P(t) = 0$ for all $t \in [-\delta, \delta]$. Therefore, $\partial^2_{q_1 q} H(te_1, 0) = 0$. □

Proof of (1). Because $\dot{Q}(0) \neq 0$, the mapping $t \mapsto Q(t)$ is an embedding near $t = 0$. Therefore, there exists $\tau > 0$ and a diffeomorphism $\varphi_0 : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $\varphi_0\big(Q(t)\big) = te_1$ for all $t \in [-\tau, \tau]$. Define

$$\Psi_0(q,p) := \big(\varphi_0(q), [d\varphi_0^{-1}]^T p\big), \quad H := \underline{H} \circ \Psi_0.$$

The Hamiltonian H satisfies (1). □

In the proofs of (2) to (5) below, let τ be the same as proof of (1). Moreover, assume that for all $t \in [-\tau, \tau]$ the given orbit $\underline{\theta}(t)$ satisfies $\underline{\theta}(t) = \big(te_1, \underline{P}(t)\big)$.
To preserve assertion (1) in the following proofs, we use only *admissible symplectomorphisms*:

Definition 1.2.3. *A fibered symplectomorphism $\Psi(q,p) = \big(\varphi(q), G(q,p)\big)$ is admissible whenever φ is identity on the segment $\big\{te_1 \mid t \in [-\tau, \tau]\big\}$, where τ is introduced in the proof of (1).*

Remark 1.2.4. *An admissible homogeneous symplectomorphism $\Psi(q,p) = \big(\varphi(q), [d\varphi^{-1}]^T p\big)$ satisfies*

$$d\varphi(te_1)e_1 = e_1, \quad \big([d\varphi^{-1}(te_1)]^T p\big)_1 = p_1, \quad \text{for all} \quad p \in (\mathbb{R}^{d+1})^*, \tag{1.2.1}$$

where $(.)_1$ denotes for the first component of a vector.

Proof of (2). Let $v(q_1) : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $v'(t) = \underline{P}_1(t)$ for all $t \in [-\tau, \tau]$. Where we used the notation $\underline{P} = (\underline{P}_1, \hat{\underline{P}}) \in \mathbb{R}^* \times (\mathbb{R}^d)^*$. Define

$$\Psi_1(q,p) := \big(q, p + du(q)\big), \quad u(q_1, \hat{q}) := v(q_1) + \hat{\underline{P}}(q_1)\hat{q}.$$

Then for all $t \in [-\tau, \tau]$, we have $\Psi_1^{-1}\big(te_1, \underline{P}(t)\big) = \big(te_1, \underline{P}(t) - du(te_1)\big) = (te_1, 0)$. □

Proof of (4). Suppose Ψ_1 is already performed. Define the vector field

$$\underline{V}(q) := \partial_p \underline{H}(q, 0).$$

Because $\underline{V}(0) = \underline{\dot{Q}}(0) \neq 0$, using tubular flow theorem (see [PMM12] Theorem 1.1), there exist an open neighborhood $\mathcal{U} \subset \mathbb{R}^{d+1}$ around 0 and a diffeomorphism $\varphi_2 : \mathcal{U} \to \mathcal{U}$ such that $(\varphi_2)_* \underline{V} = e_1$ which means that the push-forward of V by φ_2 is the constant vector field e_1 on $\mathcal{U}$. Define

$$\Psi_2(q, p) := \big(\varphi_2(q), [d\varphi_2^{-1}(q)]^T p\big), \quad H := \underline{H} \circ \Psi_2.$$

Then, we have $\partial_p H(q, 0) = e_1$ for all $q \in \mathcal{U}$ which implies that $\partial^2_{qp} H(q, 0)$ identically vanishes on $\mathcal{U}$. □

Proof of (3). Assume that we have performed $\Psi_1 \circ \Psi_2$. Recall from proof of (2) that an open neighborhood $\mathcal{U} \subset \mathbb{R}^{d+1}$ around 0 exists such that $\underline{V}(q) := \partial_p \underline{H}(q, 0) = e_1$ for all $q \in \mathcal{U}$.
Set $\underline{D}(t) := \partial^2_{\hat{p}^2} \underline{H}(te_1, 0)$. Note that since $\underline{H}$ is convex, $\partial^2_{p^2} \underline{H}(te_1, 0)$ is positive-definite for all $t \in \mathbb{R}$, so in particular $\underline{D}(t)$ is invertible for all $t \in \mathbb{R}$. Define

$$\varphi_3(q) := (q_1 + l(q_1).\hat{q}, \hat{q}),$$

where $l : \mathbb{R} \to \mathbb{R}^d$ is given by

$$l(t) := [\underline{D}(t)]^{-1} \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0). \tag{1.2.2}$$

Furthermore, define

$$\Psi_3(q, p) := (\varphi_3(q), [d\varphi_3^{-1}(q)]^T p), \quad \overline{H} := \underline{H} \circ \Psi_3.$$

Using the definition of φ_3 one can compute $[d\varphi_3^{-1}(q)]^T$ as follows

$$[d\varphi_3^{-1}(q)]^T = \begin{bmatrix} [1 + l'(q_1).\hat{q}]^{-1} & 0 \\ -[1 + l'(q_1).\hat{q}]^{-1} l(q_1) & I_d \end{bmatrix}. \tag{1.2.3}$$

Where we have denoted by l' the derivative of l. From (1.2.3) and definition of Ψ_3 we get

$$\Psi_3(te_1, p) = \big(te_1, p_1, \hat{p} - p_1 l(t)\big). \tag{1.2.4}$$

Equation (1.2.4) in above implies that

$$\partial_{\hat{p}}(\underline{H} \circ \Psi_3)(te_1, p_1, \hat{p}) = \partial_{\hat{p}} \underline{H}(te_1, p_1, \hat{p} - p_1 l(t)). \tag{1.2.5}$$

We have

$$\begin{aligned} \partial^2_{p_1 \hat{p}} \overline{H}(te_1, 0) &= \partial^2_{p_1 \hat{p}} (\underline{H} \circ \Psi_3)(te_1, 0) \\ &= \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0) - \partial^2_{\hat{p}^2} \underline{H}(te_1, 0) l(t) \qquad (1.2.6) \\ &= \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0) - \underline{D}(t)[\underline{D}(t)]^{-1} \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0) = 0. \qquad (1.2.7) \end{aligned}$$

So $\overline{H}$ satisfies (3). Note that we have obtained (1.2.6) after differentiating the right side of (1.2.5) with respect to p_1 at $p = 0$. Equation (1.2.7) is the result of replacing $\underline{D}(t)$ with $\partial^2_{\hat{p}^2} \underline{H}(te_1, 0)$, and $l(t)$ with its equivalent in (1.2.2).

Although $\overline{H}$ satisfies (3), once we performed Ψ_3 we have lost assertion (4): Note that

$$\begin{aligned}\overline{V}(q) := \partial_p \overline{H}(q,0) &= d\varphi_3^{-1}(q)\underline{V}(q)\\ &= \begin{bmatrix} [1+l'(q_1).\hat{q}]^{-1} & 0 \\ -[1+l'(q_1).\hat{q}]^{-1} l(q_1) & I_d \end{bmatrix} e_1 \\ &= ([1+l'(q_1).\hat{q}]^{-1}, 0),\end{aligned}$$

so $\partial_{\hat{q}} \overline{V}_1(te_1) = \partial^2_{\hat{q}p_1} \overline{H}(te_1, 0)$ is equal to $-l'(t)$ which does not necessarily vanish.

We regain (4) using a conformal reparametrization. Define

$$f(q) := \frac{1}{\overline{V}_1(q)}, \quad H(q,p) := f(q)\big(\overline{H}(q,p) - k\big). \tag{1.2.8}$$

For $t \in [-\tau, \tau]$, the Hamiltonian H defined above admits $(te_1, 0)$ as its 0-energy orbit. Moreover, $V(q) := \partial_p H(q, 0)$ is a unit vector field on $\mathcal{U}$, so H satisfies (4).
Since $f(te_1) = 1$ for all $t \in [-\tau, \tau]$, we can easily verify that H preserves assertion (3). □

Later in this chapter, in the proof of Proposition 1.2.8, we will see that it is impossible to regain (4) in the above proof using a further fibered symplectomorphism.

Proof of (5). Consider $\Psi_1 \circ \Psi_2 \circ \Psi_3$ as the local coordinates around 0. Based on proof of (3), the fiberwise Hessian of $\underline{H}$ has the following block form

$$\partial_{p^2} \underline{H}(te_1, 0) = \begin{bmatrix} \underline{d}_{11}(t) & 0 \\ 0 & \underline{D}(t) \end{bmatrix}, \quad t \in [-\tau, \tau], \tag{1.2.9}$$

where we have set $\underline{d}_{11}(t) := \partial^2_{p_1^2} \underline{H}(te_1, 0)$, and $\underline{D}(t) := \partial^2_{\hat{p}^2} \underline{H}(te_1, 0)$.
Because $\underline{H}$ is convex, all matrices in the set $\{\underline{D}(t) \mid t \in [-\tau, \tau]\}$ are positive-definite. Therefore, there exists a smooth curve $M(q_1) : \mathbb{R} \to GL(d)$ such that

$$\underline{D}(t) = M(t)M^T(t), \quad t \in [-\tau, \tau]. \tag{1.2.10}$$

Define

$$\varphi_4(q) := \big(q_1, M(q_1)\hat{q}\big), \quad \Psi_4(q,p) := \big(\varphi_4(q), [d\varphi_4^{-1}(q)]^T p\big), \quad \overline{H} := \underline{H} \circ \Psi_4.$$

Denote by $\dot{M}$ the derivative of M. We have

$$d\varphi_4^{-1}(q) = \begin{bmatrix} 1 & 0 \\ -M^{-1}(q_1)\dot{M}(q_1)\hat{q} & M^{-1}(q_1) \end{bmatrix},$$

so

$$[d\varphi_4^{-1}(q)]^T = \begin{bmatrix} 1 & -\hat{q}^T \dot{M}^T(q_1)[M^{-1}(q_1)]^T \\ 0 & [M^{-1}(q_1)]^T \end{bmatrix}.$$

In particular,

$$[d\varphi_4^{-1}(te_1)]^T = \begin{bmatrix} 1 & 0 \\ 0 & [M^{-1}(t)]^T \end{bmatrix}.$$

We claim that

$$\partial^2_{\hat{p}^2} \overline{H}(te_1, 0) = I. \tag{1.2.11}$$

To prove the claim (1.2.11), we write the Taylor expansion of $\overline{H}$ with respect to p-variables around $(te_1, 0)$, where $t \in [-\tau, \tau]$:

$$\begin{aligned}
\overline{H}(te_1, p) &= \underline{H} \circ \Psi_4(te_1, p) = \underline{H}\big(te_1, [d\varphi_4^{-1}(te_1)]^T p\big) \\
&= p_1 + p^T \begin{bmatrix} 1 & 0 \\ 0 & M^{-1}(t) \end{bmatrix} \begin{bmatrix} \underline{d}_{11}(t) & 0 \\ 0 & \underline{D}(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & [M^{-1}(t)]^T \end{bmatrix} p + O_3(p) \\
&= p_1 + p^T \begin{bmatrix} \underline{d}_{11}(t) & 0 \\ 0 & I_d \end{bmatrix} p + O_3(p).
\end{aligned} \tag{1.2.12}$$

Equation (1.2.12) confirms what we have asserted earlier in equation (1.2.11).

In a neighborhood of $q = 0$ we have

$$\overline{V}(q) := \partial_p \overline{H}(q, 0) = d\varphi_4^{-1}(q) \partial_p \underline{H}(q, 0) = d\varphi_4^{-1}(q) e_1 = \big(1, -M^{-1}(q_1)\dot{M}(q_1)\hat{q}\big).$$

Therefore,

$$\partial^2_{\hat{q}\hat{p}} \overline{H}(te_1, 0) = -M^{-1}(t)\dot{M}(t), \quad t \in [-\tau, \tau], \tag{1.2.13}$$

which implies that $\partial^2_{pq} \overline{H}(te_1, 0)$ does not necessarily vanish near $t = 0$. That means assertion (4) is lost again. In order to obtain (4), we apply the vertical symplectomorphism

$$\Psi_5(q, p) := \big(q, p + dg(q)\big),$$

where $g : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth function that satisfies

$$dg(te_1) = 0, \quad \partial^2_{\hat{q}^2} g(te_1) = M^{-1}(t)\dot{M}(t), \quad t \in [-\tau, \tau]. \tag{1.2.14}$$

A desired function g exists only if

$$M^{-1}(t)\dot{M}(t) \in \mathcal{S}(d) \quad \text{for all} \quad t \in [-\tau, \tau], \tag{1.2.15}$$

where $\mathcal{S}(d)$ denotes for the set of symmetric $d \times d$ real matrices. So we need to prove that a factorization $\underline{D} = M(t)M^T(t)$ exists such that (1.2.15) holds. This is proven in Lemma 1.2.5 below. We define $H := \overline{H} \circ \Psi_5$, then we have

$$\begin{aligned}
\partial^2_{\hat{p}\hat{q}} H(te_1, 0) &= \partial^2_{\hat{q}\hat{p}} \overline{H}(te_1, 0) + \partial^2_{\hat{p}^2} \overline{H}(te_1, 0) \partial^2_{\hat{q}^2} g(te_1) \\
&= -M^{-1}(t)\dot{M}(t) + I\big(M^{-1}(t)\dot{M}(t)\big) \\
&= 0.
\end{aligned} \tag{1.2.16}$$

So H satisfies (4). Note that in order to deduce (1.2.16), we used (1.2.14), (1.2.13) and (1.2.11). Because vertical symplectomorphisms has no effect on fiberwise Hessian, we have

$$\partial^2_{\hat{p}^2} H(te_1, 0) = \partial^2_{\hat{p}^2} \overline{H}(te_1, 0) = I.$$

Hence, H also satisfies (5). □

Lemma 1.2.5. *A smooth curve $M(t) : [-\tau, \tau] \to GL(d)$ exists such that for all $t \in [-\tau, \tau]$, we have $\underline{D}(t) = M(t)M^T(t)$, and $M^{-1}(t)\dot{M}(t) \in \mathcal{S}(d)$. Where for $t \in [-\tau, \tau]$, we have set $\underline{D}(t) := \partial^2_{\hat{p}^2} \underline{H}(te_1, 0)$.*

Proof. We will demonstrate that the desired curve $M(t) : [-\tau, \tau] \to GL(d)$ is the solution of the differential equation

$$\begin{cases} \dot{M}(t) = \frac{1}{2}\big(\dot{\underline{D}}(t)\big)\left[M^{-1}(t)\right]^T \\ M(0) = [\underline{D}(0)]^{\frac{1}{2}} \end{cases}, \quad t \in [-\tau, \tau]. \tag{1.2.17}$$

Suppose M is the solution of the above equation, then we have

$$M^{-1}(t)\dot{M}(t) = \frac{1}{2}M^{-1}(t)\dot{\underline{D}}(t)[M^{-1}(t)]^T. \tag{1.2.18}$$

By definition, $\dot{\underline{D}}(t)$ is symmetric for all $t \in [-\tau, \tau]$, so we conclude that

$$M^{-1}(t)\dot{\underline{D}}(t)[M^{-1}(t)]^T \in \mathcal{S}(d), \quad \text{for all} \quad t \in [-\tau, \tau]. \tag{1.2.19}$$

Therefore, based on (1.2.18) and (1.2.19), it is obvious that $M^{-1}(t)\dot{M}(t)$ is symmetric for all $t \in [-\tau, \tau]$.
It remains to prove that $\underline{D}(t) = M(t)M^T(t)$. Since $M^{-1}(t)\dot{M}(t) \in \mathcal{S}(d)$, we have

$$M^{-1}(t)\dot{M}(t) = [\dot{M}(t)]^T[M^{-1}(t)]^T. \tag{1.2.20}$$

Multiplying $M(t)$ from the left and $[M(t)]^T$ from the right to (1.2.20) yields the following equation

$$\dot{M}(t)[M(t)]^T = M(t)[\dot{M}(t)]^T. \tag{1.2.21}$$

Based on (1.2.17) and (1.2.21)

$$\begin{aligned} \dot{\underline{D}}(t) &= 2\dot{M}(t)[M(t)]^T \\ &= \dot{M}(t)[M(t)]^T + M(t)[\dot{M}(t)]^T, \quad t \in [-\tau, \tau]. \end{aligned} \tag{1.2.22}$$

Equation (1.2.22) and the initial condition $M(0) = [\underline{D}(0)]^{\frac{1}{2}}$ imply that $\underline{D} = M(t)M^T(t)$. □

Here we aim to adjust the proofs of (1) to (5) with the notation that we have used in the statement of Theorem 1.2.1. Define

$$\Psi^1 := \Psi_3 \circ \Psi_0 \circ \Psi_1 \circ \Psi_2, \quad \Psi^2 := \Psi_4 \circ \Psi_5, \quad \Psi := \Psi^1 \circ \Psi^2,$$

and

$$\varphi^1 := \varphi_0 \circ \varphi_1 \circ \varphi_2 \circ \varphi_3, \quad \varphi^2 := \varphi_4 \circ \varphi_5.$$

Moreover, define $z(q) := f \circ \varphi^2(q)$, where

$$f(q) := \frac{1}{\partial_{p_1}\big(\underline{H} \circ \Psi_0 \circ \Psi_1 \circ \Psi_2\big)(q, 0)}.$$

Based on the proofs of (1) to (5) in above, there exists $\delta > 0$ such that for all $t \in [-\delta, \delta]$, $\big(Q(t), P(t)\big) := \Psi^{-1}(\underline{\theta})$ and the Hamiltonian H defined as follows

$$\begin{aligned} H(q,p) := \big[f(q)\big((\underline{H} \circ \Psi^1)(q,p) - k\big)\big] \circ \Psi^2 &= (f \circ \varphi^2)(q)\big((\underline{H} \circ \Psi^1 \circ \Psi^2) - k\big) \\ &= z(q)\big(\underline{H}(q,p) \circ \Psi - k\big) \end{aligned}$$

satisfy assertions (1) to (5) of the Theorem 1.2.1.

1.2.2 The normal form claimed by Figalli and Rifford

Now we restate the Lemma C1 of [FR15] in Proposition 1.2.6 below. Proposition 1.2.8 that follows in this section shows that Proposition 1.2.6 is false.

Proposition 1.2.6. *Assume that $\underline{H}(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth convex Hamiltonian and $\underline{\theta}(t) = \big(\underline{Q}(t), \underline{P}(t)\big)$ is a given orbit of Hamiltonian vector field of $\underline{H}$ such that $\dot{\underline{Q}}(0) \neq 0$. There exist a real number $\delta > 0$, and a homogeneous symplectomorphism $\Psi(q,p) : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$ such that for all $t \in [-\delta, \delta]$ we have*

(1) $Q(t) = te_1, \quad e_1 = (1, 0_d)$

(2) $P(t) = 0$

(3) $\partial^2_{p_1\hat{p}} H(te_1, 0) = 0$

(4) $\partial^2_{qp} H(te_1, 0) = 0$

(5) $\partial^2_{\hat{p}^2} H(te_1, 0) = I$,

where $H := \underline{H} \circ \Psi$ and $\big(Q(t), P(t)\big) := \Psi^{-1}\big(\underline{\theta}(t)\big)$.

Remark 1.2.7. *We have replaced*

$(2)'$ $P(t) = e_1$.

in the original statement of Lemma C1 in [FR15] with (2) above. More explanation is given below.

Let us first note the differences between the above proposition —which is equivalent to Lemma C1 of [FR15]— and Theorem 1.2.1. Besides homogeneous symplectomorphisms, the statement of the alternative normal form allows us to use vertical symplectomorphisms, in addition, the alternative normal form permits to reparametrize the Hamiltonian vector field with a function that only depends on q-variable. At the other hand, in Proposition 1.2.6, our only tool to maintain the assertions is homogeneous symplectic changes of coordinates.

A homogeneous symplectomorphism preserves the zero section, so in general, both (2) and $(2)'$ are not obtainable by an admissible homogeneous symplectic change of coordinates. Furthermore, given a regular orbit $\underline{\theta}(t) = \big(\underline{Q}(t), \underline{P}(t)\big)$, an admissible homogeneous symplectomorphism preserves the first coordinates of $\underline{P}(t)$, namely $\underline{P}_1(t)$. To see the reason of that, it is enough to recall equation (1.2.1):

$$([d\varphi^{-1}(te_1)]^T p)_1 = p_1, \quad \text{for all}\, p \in (\mathbb{R}^{d+1})^*.$$

So once $\underline{P}_1(t)$ is given, it cannot be changed by an admissible homogeneous symplectic change of coordinates.

Acquiring (2) or $(2)'$ would be easy using vertical symplectomorphisms. See proof of (2) of Theorem 1.2.1. Applying vertical symplectomorphisms is also necessary for assertions (4) and (5) to hold simultaneously. Look at the proof of assertion (5) of Theorem 1.2.1.

As we will see in the next subsection, having $(2)'$ instead of (2) in similar normal forms is convenient whenever we are working with homogeneous Hamiltonians; Otherwise, we prefer to work with normal forms that are satisfying (2) instead of $(2)'$. As we already explained, associated to our methods, there is not much of a difference between proving (2) or $(2)'$.

Above all, the major issue of Proposition 1.2.6 is that whenever the map $t \mapsto \partial^2_{\hat{q}p_1}\underline{H}(te_1, 0)$ is not identically equal to zero it cannot be converted to the null function via fibered symplectomorphisms that are preserving (1),(2) and (3). We state this more precisely in the following Proposition.

Proposition 1.2.8. *Let $\underline{H}(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ be a given smooth convex Hamiltonian. Assume that for some $\delta > 0$, $\underline{\theta}(t) = (te_1, 0)$ where $t \in [-\delta, \delta]$, is an orbit segment of the Hamiltonian vector field of $\underline{H}$. Suppose that we have*

$$\partial^2_{p_1\hat{p}}\underline{H}(te_1, 0) = 0, \quad \textit{for all} \quad t \in [-\delta, \delta]. \tag{1.2.23}$$

Furthermore, assume that $\partial^2_{\hat{q}p_1}\underline{H}(0) \neq 0$. Then there is not exists an admissible fibered symplectic change of coordinates $\Omega(q,p) : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$, $\Omega^{-1}(te_1, 0) = (te_1, 0)$, such that $H := \underline{H} \circ \Omega$ preserves equation (1.2.23) (i.e. $\partial^2_{p_1\hat{p}}H(te_1, 0) = 0$ for all $t \in [-\delta, \delta]$), and satisfies $\partial^2_{\hat{q}p_1}H(0) = 0$.

Proof of Proposition 1.2.8

First, we show that the value of $\partial^2_{\hat{q}p_1}\underline{H}(0)$ is invariant under any vertical symplectic change of coordinates of the form that follows

$$\Omega(q,p) = \big(q, p + dg(q)\big), \quad dg(te_1) = 0. \tag{1.2.24}$$

Note that condition $dg(te_1) = 0$ is necessary for Ω to satisfy $\Omega^{-1}(te_1, 0) = (te_1, 0)$. Since we have

$$\partial_{p_1}\big(\underline{H} \circ \Omega\big)(q,p) = \partial_{p_1}\underline{H}\big(q, p + dg(q)\big),$$

then for all $t \in [-\delta, \delta]$, we can write

$$\partial^2_{\hat{q}p_1}\big(\underline{H} \circ \Omega\big)(te_1, 0) = \partial^2_{\hat{q}p_1}\underline{H}(te_1, 0) + \sum_{i=1}^{n} \partial^2_{p_i p_1}\underline{H}(te_1, 0)\partial^2_{\hat{q}q_i}g(te_1). \tag{1.2.25}$$

Recall from the assumptions of the Proposition 1.2.8 that $\partial^2_{\hat{p}p_1}\underline{H}(te_1, 0) = 0$, so we can rewrite equation (1.2.25) as

$$\partial^2_{\hat{q}p_1}\big(\underline{H} \circ \Omega\big)(te_1, 0) = \partial^2_{\hat{q}p_1}\underline{H}(te_1, 0) + \partial^2_{p_1^2}\underline{H}(te_1, 0)\partial^2_{\hat{q}q_1}g(te_1). \tag{1.2.26}$$

Because $\partial_{\hat{q}}g(te_1) = 0$ for all $t \in [-\delta, \delta]$, we obtain

$$\partial^2_{q_1\hat{q}}g(te_1) = 0, \quad t \in [-\delta, \delta]. \tag{1.2.27}$$

After inserting (1.2.27) into (1.2.26) we get

$$\partial^2_{\hat{q}p_1}\big(\underline{H} \circ \Omega\big)(te_1, 0) = \partial^2_{\hat{q}p_1}\underline{H}(te_1, 0), \quad t \in [-\delta, \delta].$$

We just proved that, for all $t \in [-\delta, \delta]$, the value of $\partial^2_{\hat{q}p_1}\underline{H}(te_1, 0)$ is invariant under vertical symplectic change of coordinates of the form (1.2.24). So in particular, if the value of $\partial^2_{\hat{q}p_1}\underline{H}(0)$ is non-zero it cannot be changed by vertical symplectomorphisms of the form (1.2.24).

To complete the proof of Proposition 1.2.8, we also need to show that $\partial^2_{\hat{q}p_1}\underline{H}(0)$ does not vanish after performing any admissible homogeneous symplectic change of coordinates that preserves (1.2.23). To do so, first we need to prove the following lemma.

Lemma 1.2.9. *Consider an admissible homogeneous symplectomorphism*

$$\Omega(q,p) = \big(\varphi(q), [d\varphi^{-1}(q)]^T p\big)$$

that preserves (1.2.23). Then $d\varphi(te_1)$ has the following block form

$$d\varphi(te_1) = \begin{bmatrix} 1 & 0_d \\ 0_d & * \end{bmatrix},$$

for all $t \in [-\delta, \delta]$, where δ is the same constant as in Proposition 1.2.8

Proof. Since Ω is admissible, by (1.2.1) we have

$$[d\varphi(te_1)]e_1 = e_1, \quad \big([d\varphi^{-1}(te_1)]^T p\big)_1 = p_1 \quad \text{for all} \quad p \in (\mathbb{R}^{d+1})^*.$$

Therefore, $[d\varphi^{-1}(te_1)]^T$ must have the triangular block form

$$[d\varphi^{-1}(te_1)]^T = \begin{bmatrix} 1 & 0_d \\ b(t) & B(t) \end{bmatrix}, \tag{1.2.28}$$

for some $b(t) : [-\delta, \delta] \to \mathbb{R}^d$, and $B(t) : [-\delta, \delta] \to GL(d)$. Note that

$$d\varphi(te_1) = \begin{bmatrix} 1 & -b^T(t)[B^{-1}(t)]^T \\ 0 & [B^{-1}(t)]^T \end{bmatrix}. \tag{1.2.29}$$

If we set $H := \underline{H} \circ \Omega$, $\underline{D}(t) := \partial^2_{\hat{p}^2}\underline{H}(te_1, 0)$, and $\underline{d}_{11}(t) := \partial^2_{p_1^2}\underline{H}(te_1, 0)$, then we can write

$$\begin{aligned}
\partial^2_{\hat{p}^2} H(te_1, 0) = \partial^2_{\hat{p}^2}\big(\underline{H} \circ \Omega\big)(te_1, 0) &= [d\varphi^{-1}(te_1)]\partial^2_{\hat{p}^2}\underline{H}(te_1, 0)[d\varphi^{-1}(te_1)]^T \\
&= \begin{bmatrix} 1 & [b(t)]^T \\ 0 & [B(t)]^T \end{bmatrix} \begin{bmatrix} \underline{d}_{11} & 0 \\ 0 & \underline{D}(t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b(t) & B(t) \end{bmatrix} \\
&= \begin{bmatrix} * & [b(t)]^T \underline{D}(t) B(t) \\ [B(t)]^T \underline{D}(t) b(t) & [B(t)]^T \underline{D}(t) \end{bmatrix}.
\end{aligned} \tag{1.2.30}$$

By assumption, Ω preserves (1.2.23), so (1.2.30) implies that

$$[b(t)]^T \underline{D}(t) B(t) = 0 \Rightarrow b(t) = 0. \tag{1.2.31}$$

In the above equation, note that $\underline{D}(t)$ is invertible because $\underline{H}$ is convex; Moreover, after recalling equation (1.2.28) and the fact that φ is a diffeomorphism, we see that $B(t)$ is invertible as well. Finally, from (1.2.31) and (1.2.28) we conclude that

$$[d\varphi^{-1}(te_1)]^T = \begin{bmatrix} 1 & 0 \\ 0 & B(t) \end{bmatrix} \Rightarrow d\varphi(te_1) = \begin{bmatrix} 1 & 0 \\ 0 & [B^{-1}(t)]^T \end{bmatrix}. \tag{1.2.32}$$

□

We continue the proof of Proposition 1.2.8. Assume that $\Omega = \big(\varphi(q), [d\varphi^{-1}(q)]^T p\big)$ is an admissible homogeneous symplectomorphism that preserves (1.2.23). Set $\underline{V}(q) := \partial_p \underline{H}(q, 0)$, and let V be the push-forward of $\underline{V}$ by Ω. We have

$$\underline{V}(\varphi(q)) = d\varphi(q)V(q). \tag{1.2.33}$$

Let $a(q)$ be the first coordinate of $\varphi(q)$. Moreover, denote by V_1 and $\underline{V}_1$ the first coordinates of V and $\underline{V}$ respectively. From equation (1.2.33), we have

$$\underline{V}_1\big(\varphi(q)\big) = \partial_q a(q)V(q).$$

Therefore, if we set $\partial_{\hat{q}}\varphi(te_1) =: Z(t)$ we can write the following

$$\partial_{\hat{q}}\underline{V}_1(te_1)Z(t) = \partial^2_{\hat{q}q}a(te_1)V(te_1) + \partial_q a(te_1)\partial_{\hat{q}}V(te_1). \tag{1.2.34}$$

Note that for all $t \in [-\delta, \delta]$, we have

$$V(te_1) = e_1. \tag{1.2.35}$$

Besides, because of Lemma 1.2.9, we know that $d\varphi$ must have a block form similar to (1.2.32) which gives $\partial_q a(te_1) = e_1$, and in particular $\partial_{\hat{q}}a(te_1) = 0$, for all $t \in [-\delta, \delta]$. So we have

$$\partial^2_{q_1\hat{q}}a(te_1) = \lim_{\epsilon\to 0}\frac{\partial_{\hat{q}}a((t+\epsilon)e_1) - \partial_{\hat{q}}a(te_1)}{\epsilon} = 0, \quad t \in [-\delta, \delta]. \tag{1.2.36}$$

Based on equations (1.2.34), (1.2.35), and (1.2.36) we can write

$$\begin{aligned}\partial_{\hat{q}}\underline{V}_1(te_1)Z(t) &= \partial^2_{\hat{q}q}a(te_1)e_1 + e_1\partial_{\hat{q}}V(te_1) = \partial^2_{\hat{q}q_1}a(te_1) + \partial_{\hat{q}}V_1(te_1)\\ &= \partial_{\hat{q}}V_1(te_1).\end{aligned} \tag{1.2.37}$$

Because $Z(t)$ is invertible by its definition, from equation (1.2.37) we conclude that if $\partial_{\hat{q}}\underline{V}_1(0) = \partial^2_{\hat{q}p_1}\underline{H}(0) \neq 0$ then $\partial_{\hat{q}}V_1(0) = \partial^2_{\hat{q}p_1}H(0) \neq 0$. That means whenever the value of $\partial^2_{\hat{q}p_1}\underline{H}(0)$ is non-zero it will not vanish after an admissible homogeneous symplectic change of coordinates.

1.3 Normal form for homogeneous Hamiltonians

We prove a normal form for homogeneous Hamiltonians. See Theorem 1.4.1 below. Then, we show that Theorem 1.4.1 implies the Li-Nirenberg's normal form which we represent as Corollary 1.3.2. Our purpose in this section is to remove the confusion that exists in the literature between Li-Nirenberg's normal form and a similar normal form for convex Hamiltonians.

Theorem 1.3.1 (Normal form for homogeneous Hamiltonians)**.** *Assume that $\underline{H} : T^*\mathbb{R}^{d+1} \to \mathbb{R}^+$ is a positively β-homogeneous Hamiltonian where $\beta \geq 1$. Suppose that $\underline{H}$ is convex and smooth on $p \neq 0$. Let $\underline{\theta}(t) = \big(\underline{Q}(t), \underline{P}(t)\big)$ be a given orbit of Hamiltonian vector field of $\underline{H}$ such that $\dot{\underline{Q}}(0) \neq 0$ and $\underline{H}\big(\underline{\theta}(t)\big) = k$. There exist $\delta > 0$, and a smooth homogeneous symplectomorphism $\Psi(q,p) : T^*M \to T^*M$ such that for $H := \underline{H} \circ \Psi$ and $\big(Q(t), P(t)\big) := \Psi^{-1}\big(\underline{Q}(t), \underline{P}(t)\big)$ the following assertions are true for all $t \in [-\delta, \delta]$*

(1) $Q(t) = te_1, \quad e_1 = (1, 0_d)$

(2) $P(t) = (\beta k, 0_d)$

(3) $\partial^2_{p_1\hat{p}}H(te_1, \beta k, 0_d) = 0$

(4) $\partial^2_{qp}H(te_1, \beta k, 0_d) = 0.$

Proof of (1). This proof is the same as proof of (1) of Theorem 1.2.1.
Since we have $\dot{\underline{Q}}(0) \neq 0$, the mapping $t \mapsto Q(t)$ is an embedding near $t = 0$. So there exist $\tau > 0$, and a diffeomorphism $\varphi_0 : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $\varphi_0\big(\underline{Q}(t)\big) = te_1$ where $t \in [-\tau, \tau]$. So to have (1), it is enough to apply the homogeneous symplectic change of coordinates Ψ_0 defined as follows

$$\Psi_0(q,p) := \big(\varphi_0(q), [d\varphi^{-1}(q)]^T p\big).$$

□

In the proofs of (2) to (5), we assume that for all $t \in [-\tau, \tau]$ the given orbit $\underline{\theta}(t)$ satisfies $\underline{\theta}(t) = \big(te_1, \underline{P}(t)\big)$. Furthermore, in order to preserve (1), we only use admissible homogeneous symplectic changes of coordinates. See Definition 1.2.3 to recall what we mean by an admissible homogeneous symplectomorphism.

Note that for a given $\eta \in [-\tau, \tau]$, $\underline{P}(\eta)$ does not vanish. Otherwise, by homogeneity of H we have $H\big(\eta e_1, \underline{P}(\eta)\big) = 0$ which contradicts with the assumption that H admits only positive values. Let $\underline{P}_1(t)$ denote the first component of $\underline{P}(t)$, then for all $t \in [-\tau, \tau]$ we have

$$\begin{aligned} \underline{P}_1(t) &= \underline{P}(t).e_1 \\ &= \underline{P}(t).\partial_p \underline{H}\big(te_1, \underline{P}(t)\big) \\ &= \beta \underline{H}\big(te_1, \underline{P}(t)\big) && (1.3.1) \\ &= \beta k. && (1.3.2) \end{aligned}$$

where we have conclude (1.3.1) by Euler's theorem for homogeneous functions (Theorem 1.1.5). Because β is assumed to be greater or equal than 1 and k is nonzero, equation (1.3.2) implies that $\underline{P}_1(t)$ does not vanish for all $t \in [-\tau, \tau]$.

Proof of (2) and (3) . We define

$$\varphi_1(q) := \big(q_1 + l(q_1).\hat{q}, \hat{q}\big), \quad \Psi_1(q,p) := \big(\varphi_1(q), [d\varphi_1^{-1}(q)]^T p\big),$$

for $l(q_1) : \mathbb{R} \to \mathbb{R}^d$ that is given by

$$l(t) := \frac{\underline{\hat{P}}(t)}{\underline{P}_1(t)}, \quad t \in [-\tau, \tau].$$

Where we have used the notation $\underline{P} = \big(\underline{P}_1, \underline{\hat{P}}\big) \in \mathbb{R} \times \mathbb{R}^d$. Right above this proof we have reasoned that why $\underline{P}_1(t) \neq 0$ for all $t \in [-\tau, \tau]$.
By definition of φ_1, we have $[d\varphi_1^{-1}(q_1, 0)]^T = \begin{bmatrix} 1 & 0_d \\ -l(q_1) & I_d \end{bmatrix}$ which implies that

$$\Psi_1(q_1, 0_d, p_1, \hat{p}) = (q_1, 0_d, p_1, \hat{p} - p_1 l(q_1)).$$

Therefore, because $\underline{\hat{P}}(t) - \underline{P}_1(t) l(t) = 0$, we have

$$\Psi_1(te_1, \underline{P}(t)) = (te_1, \underline{P}_1(t), 0_d), \quad t \in [-\tau, \tau]. \tag{1.3.3}$$

Equation (1.3.2) gives $\underline{P}_1(t) = \beta k$ for all $t \in [-\tau, \tau]$. Hence, we can rewrite (1.3.3) as

$$\Psi_1(te_1, \underline{P}(t)) = (te_1, \beta k, 0_d), \quad t \in [-\tau, \tau],$$

and that finishes the proof of (2).

In order to obtain (3), there is no need for a further symplectic change of coordinates. We will show that $H := \underline{H} \circ \Psi_1$ automatically satisfies (3). Euler's theorem implies that the mapping $p \mapsto \partial_{\hat{p}} H(te_1, p)$ is positively $(\beta - 1)$-homogeneous. Therefore, for all $t \in [-\tau, \tau]$ we have

$$\begin{aligned} \partial^2_{p_1 \hat{p}} H(te_1, \underline{P}_1, 0_d) &= \lim_{\epsilon \to 0} \frac{\partial_{\hat{p}} H(te_1, \underline{P}_1 + \epsilon, 0_d) - \partial_{\hat{p}} H(te_1, \underline{P}_1, 0_d)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{(\underline{P}_1 + \epsilon)^{\beta-1} \partial_{\hat{p}} H(te_1, e_1) - \underline{P}_1^{\beta-1} \partial_{\hat{p}} H(te_1, e_1)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\big((\underline{P}_1 + \epsilon)^{\beta-1} - \underline{P}_1^{\beta-1}\big) \frac{d}{dt} \hat{Q}(t)}{\epsilon} = 0. \end{aligned}$$

In above, $\hat{Q}$ denotes for the second component in the decomposition $Q = \big(Q_1, \hat{Q}\big) \in \mathbb{R} \times \mathbb{R}^d$. Note that $Q(t) = te_1$ where $t \in [-\tau, \tau]$, so $\frac{d}{dt}\hat{Q}(t) = 0$ for all $t \in [-\tau, \tau]$. □

Proof of (4). Suppose Ψ_1 is already performed. Consider the vector field $\underline{V}(q) := \partial_p \underline{H}(q, \beta k, 0_d)$. Because $\underline{V}(0) = \dot{\underline{Q}}(0) \neq 0$, by tubular flow theorem there exists an open set $\mathcal{U} \subset \mathbb{R}^{d+1}$ and a diffeomorphism $\varphi_2 : \mathcal{U} \to \mathcal{U}$ such that $V := (\varphi_2)_* \underline{V}$ is the unit vector field e_1 on $\mathcal{U}$. Therefore, after defining

$$\Psi_2(q, p) := \big(\varphi_2(q), [d\varphi_2^{-1}(q)]^T p\big), \quad H := \underline{H} \circ \Psi_2,$$

we have

$$\partial_p H(q, \beta k, 0_d) = e_1, \quad \text{for all} \quad q \in \mathcal{U}. \tag{1.3.4}$$

Differentiating (1.3.4) with respect to q finishes the proof. □

The particular case of $\beta = 2$ of Theorem 1.3.1 implies the Li-Nirenberg normal form.

Corollary 1.3.2 (Li-Nirenberg normal form)**.** *Assume that $\big(\underline{Q}(t), \dot{\underline{Q}}(t)\big)$ is a geodesic of a given Finsler metric $\underline{L}(q, v) : T\mathbb{R}^{d+1} \to \mathbb{R}^+$. That means $\big(\underline{Q}(t), \dot{\underline{Q}}(t)\big)$ solves the Euler-Lagrange equation*

$$\frac{d}{dt}\partial_v \underline{L}\big(\underline{Q}(t), \dot{\underline{Q}}(t)\big) = \partial_q \underline{L}\big(\underline{Q}(t), \dot{\underline{Q}}(t)\big). \tag{1.3.5}$$

There exists $\delta > 0$, and a non-singular changes of coordinates

$$\begin{aligned} &\Xi(q, v) : T\mathbb{R}^{d+1} \to T\mathbb{R}^{d+1}, \\ &(q, v) \mapsto \big(\varphi(q), d\varphi(q)v\big), \quad \varphi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \quad \textit{is a diffeomorphism,} \end{aligned}$$

such that for L defined as $L := \underline{L} \circ \Xi$ and $\big(Q(t), \dot{Q}(t)\big) := \Xi^{-1}\big(\underline{Q}(t), \dot{\underline{Q}}(t)\big)$ the following assertions are true for all $t \in [-\delta, \delta]$

(a) $Q(t) = te_1, \quad \dot{Q}(t) = e_1$
(b) $\partial_q L(te_1, e_1) = 0$
(c) $\partial_v L(te_1, e_1) = (c, 0_d),$ *for some constant* $c \in \mathbb{R}^+$
(d) $\partial^2_{v_1 \hat{v}} L(te_1, e_1) = 0$
(e) $\partial^2_{vq} L(te_1, e_1) = 0.$

The Euler-Lagrange equations for $\underline{L}$ and $\underline{L}^2$ are the same, so $\underline{L}$ and $\underline{L}^2$ share the same geodesics. Furthermore, Lemma 1.1.8 implies that the dual Hamiltonian of $\underline{L}^2$ is 2-homogeneous. In this way, we can apply the case $\beta = 2$ of Theorem 1.3.1 to conclude the above corollary.

Proof. Assume that $\underline{H}$ is the dual Hamiltonian with respect to $\underline{L}^2$, and $\big(\underline{Q}(t), \underline{P}(t)\big)$ is the orbit in the phase space corresponded to $\big(\underline{Q}(t), \dot{\underline{Q}}(t)\big)$. By Theorem 1.3.1, there exists a real number $\delta > 0$ and a homogeneous symplectomorphism $\Psi(q, p) = \big(\varphi(q), [d\varphi^{-1}(q)]^T p\big)$ such that $\varphi\big(\underline{Q}(t)\big) = te_1$ for all $t \in [-\delta, \delta]$, and $[d\varphi^{-1}(te_1)]^T \underline{P}(t) = (c', 0_d) =: P(t)$ for some constant $c' \in \mathbb{R}^+$. So if we define

$$L^2 := \underline{L}^2 \circ \Xi, \quad \Xi(q, v) = \big(\varphi(q), d\varphi(q)v\big).$$

We have

$$\partial_v L^2\big(te_1, e_1\big) = P(t) = (c', 0_d).$$

Since $\partial_v L^2 = 2L\partial_v L$ and L admits only positive values we proved part (c) of the corollary. From equation (1.3.5) and (c) we immediately conclude part (b).

Part (d) would be a consequence of (c) and homogeneity of L. By Euler's theorem for homogeneous functions (also look at the Remark 1.1.6), since $L(q,v)$ is positively 1-homogeneous, the mapping $v \mapsto \partial_{\hat{v}} L(q,v)$ is positively homogeneous of degree zero, so we have

$$\begin{aligned}\partial^2_{v_1\hat{v}} L(te_1, e_1) &= \lim_{\epsilon\to 0} \frac{\partial_{\hat{v}} L(te_1, 1+\epsilon, 0_d) - \partial_{\hat{v}} L(te_1, 1, 0_d)}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{\partial_{\hat{v}} L(te_1, 1, 0_d) - \partial_{\hat{v}} L(te_1, 1, 0_d)}{\epsilon} = 0.\end{aligned}$$

It remains to prove (e). Define $H := \underline{H} \circ \Psi$, then because $\partial_q L^2(q,v) = \partial_q H(q,p)$ we are able to obtain the following

$$\partial^2_{vq} L^2(q,v) = \partial^2_{pq} H(q,p) \partial^2_{v^2} L^2(q,v). \tag{1.3.6}$$

By definition of a Finsler metric, L^2 is convex. So in particular, $\partial^2_{v^2} L^2(q,v)$ is invertible for all $(q,v) \in T\mathbb{R}^{d+1}$. Hence, from equation (1.3.6) and part (4) of Theorem 1.3.1 we conclude that

$$\partial^2_{vq} L^2(te_1, e_1) = 0 \quad \text{for all} \quad t \in [-\delta, \delta]. \tag{1.3.7}$$

Note that L admits only positive values; Moreover, we have $\partial^2_{vq} L^2 = 2\partial_v L \partial_q L + 2L\partial^2_{qv} L$, and $\partial_q L(te_1, e_1) = 0$ for all $t \in [-\delta, \delta]$. Therefore, equation (1.3.7) implies that $\partial^2_{vq} L(te_1, e_1) = 0$ for all $t \in [-\delta, \delta]$. □

1.4 Normal form for non-convex Hamiltonians

Our motivation to prove the normal form given in Theorem 1.4.1 below lies in the purpose of generalizing the perturbation theorem (Theorem D) for non-convex Hamiltonians. We will use Theorem 1.4.1 in Section 2.2.1 during the proof of Theorem 3. Besides, Theorem 1.4.1 has direct applications in the proof of Theorem 4. See the proof of Lemma 3.2.3 in Section 3.2.1 for example.

Theorem 1.4.1 (Normal form for non-convex Hamiltonians)**.** *Suppose $\underline{H}(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth Hamiltonian. Consider $\underline{\theta}(t) = \big(\underline{Q}(t), \underline{P}(t)\big)$ as an orbit of Hamiltonian vector field of $\underline{H}$ such that $\underline{H}(\underline{\theta}) = k$. Moreover, assume that $\underline{\theta}(0) \notin \Gamma_{\underline{H}}$. There exist $\delta > 0$, a smooth fibered symplectic diffeomorphism $\Psi : T^*\mathbb{R}^{d+1} \to T^*\mathbb{R}^{d+1}$, and a smooth function $z(q) : \mathbb{R}^{d+1} \to \mathbb{R}$ such that $\big(Q(t), P(t)\big) := \Psi^{-1}(\underline{\theta})$ is an orbit of Hamiltonian vector field of $H := z(q)(\underline{H} \circ \Psi - k)$, and for all $t \in [-\delta, \delta]$ we have*

(1) $Q(t) = te_1$,

(2) $P(t) = 0$

(3) $\partial^2_{p_1\hat{p}} H(te_1, 0) = 0$,

(4) $\partial^2_{qp} H(te_1, 0) = 0$,

(5) $\partial^2_{\hat{p}^2} H(te_1, 0) = D$,

where D is a constant diagonal matrix with only 1 and -1 entries.

Remark 1.4.2. *Assertions (1) and (2) yield*

(6) $\partial^2_{q_1 q} H(te_1, 0) = 0, \quad t \in [-\delta, \delta]$.

Proofs of (1),(2) and (4) are the same as the proofs of similar assertions in Theorem 1.2.1. However, to avoid ambiguities, in the proof of Theorem 1.4.1 we do not refer to the proof of Theorem 1.2.1.

Note that Theorem 1.4.1 is in fact a generalization of the alternative normal form. To see the reason, it is enough to compare the statements of the two normal forms and recall that a convex Hamiltonian $\underline{H}$ is iso-energetically non-degenerate at $\underline{\theta}(t_0)$ if $\underline{\theta}(t_0)$ is a neat point, where $\underline{\theta}(t)$ is assumed to be an orbit of Hamiltonian vector field of $\underline{H}$.

Proof of (1). The assumption $\underline{\theta}(0) \notin \Gamma_{\underline{H}}$ implies that $\partial_p \underline{H}\big(\underline{\theta}(0)\big) = \dot{\underline{Q}}(0) \neq 0$. Therefore, since the mapping $t \mapsto Q(t)$ is an embedding near $t = 0$, there exists $\tau > 0$, and a diffeomorphism $\varphi_0(q) : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that $\varphi_0\big(\underline{Q}(t)\big) = te_1$ for all $t \in [-\tau, \tau]$. Consider the homogeneous symplectomorphism as follows

$$\Psi_0(q,p) := \big(\varphi_0(q), [d\varphi_0^{-1}(q)]^T p\big).$$

The Hamiltonian H defined as $H := \underline{H} \circ \Psi_0$ satisfies (1). □

In proofs of (2) to (5), assume that $\underline{\theta}(t)$ is given as $\big(te_1, \underline{P}(t)\big)$, where $t \in [-\tau, \tau]$.

Proof of (2). Define $u : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ as $u(q_1, \hat{q}) := v(q_1) + \hat{\underline{P}}(q_1).\hat{q}$ where $v(t) : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $v' = \underline{P}_1$. Consider the vertical symplectic change of coordinates

$$\Psi_1(q,p) := \big(q, p + du(q)\big).$$

Because $du(te_1) = \underline{P}(t)$, we have $\Psi_1^{-1}\big(te_1, \underline{P}(t)\big) = (te_1, 0)$. That means $H := \underline{H} \circ \Psi_1$ satisfies (2). □

Note that we are able to prove assertions (1) and (2) using the assumption $\partial_p \underline{H}\big(\underline{\theta}(0)\big) \neq 0$. Assuming that $\partial_p \underline{H}\big(\underline{\theta}(0)\big)$ does not vanish is weaker in compare with the assumption $\underline{\theta}(0) \notin \Gamma_{\underline{H}}$, and we have assumed the latter in the statement of Theorem 1.4.1. In Section 3.2.2, the proof of Lemma 3.2.4, we apply assertions (1) and (2) of the normal form around a point with non-zero velocity at which a non-convex Hamiltonian is not necessarily fiberwise iso-energetically non-degenerate. At the other hand, the weaker assumption $\partial_p \underline{H}\big(\underline{\theta}(0)\big) \neq 0$ is not enough to prove assertions (3) and (5); That would be obvious once we give a proof for (3) and (5).

Proof of (4). Assume that Ψ_1 which we introduced in the proof of (2) is the local coordinates around $\underline{\theta}(0)$. Define the vector field

$$\underline{V}(q) := \partial_p \underline{H}(q, 0).$$

Because $\dot{\underline{Q}}(0)$ is non-zero, by tubular flow theorem there exist an open neighborhood $\mathcal{U} \subset \mathbb{R}^{d+1}$ around $q = 0$, and a diffeomorphism $\varphi_2(q) : \mathcal{U} \to \mathcal{U}$ such that $(\varphi_2)_* \underline{V}$ is the constant vector field e_1 for all $q \in \mathcal{U}$. Define

$$\Psi_2(q,p) := \big(\varphi_2(q), [d\varphi_2^{-1}(q)]^T p\big), \quad H := \underline{H} \circ \Psi_2.$$

Then we have

$$\partial_p H(q, 0) = e_1, \quad \text{for all} \quad q \in \mathcal{U}. \tag{1.4.1}$$

Differentiating (1.4.1) with respect to q implies (4). □

Proof of (3). Take $\Psi_2 \circ \Psi_1$ as the local coordinates around $\underline{\theta}(0)$. Set

$$\underline{D}(t) := \partial^2_{p^2} \underline{H}(te_1, 0), \quad t \in [-\tau, \tau]. \tag{1.4.2}$$

After recalling the definition of Γ_H in equation (2), because $\underline{\theta}(0) \notin \Gamma_{\underline{H}}$ and $\partial_p \underline{H}\big(\underline{\theta}(0)\big) = e_1$, it would be easy to verify that $\underline{D}(0)$ is invertible. In consequence, we are able to choose $\tau_1 \in (0,\tau)$ such that $\underline{D}(t) \in GL(d)$ for all $t \in [-\tau_1, \tau_1]$. Define

$$\varphi_3(q) := \big(q_1 + l(q_1).\hat{q}, \hat{q}\big), \quad \Psi_3(q,p) := \big(\varphi_3(q), [d\varphi_3^{-1}(q)]^T p\big), \quad H := \underline{H} \circ \Psi_3,$$

where $l(q_1) : \mathbb{R} \to \mathbb{R}^d$ is given as follows

$$l(t) = [\underline{D}(t)]^{-1} \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0), \quad t \in [-\tau_1, \tau_1].$$

Afterwards, we have

$$[d\varphi_3^{-1}(te_1)]^T = \begin{bmatrix} [1 + l'(q_1).\hat{q}]^{-1} & 0 \\ -[1 + l'(q_1).\hat{q}]^{-1} l(q_1) & I_d \end{bmatrix},$$

which yields

$$\Psi_3(te_1, 0) = \big(te_1, p_1, \hat{p} - p_1 l(t)\big). \tag{1.4.3}$$

From (1.4.3) we conclude that $\partial_{\hat{p}}(\underline{H} \circ \Psi_3)(te_1, p_1, \hat{p}) = \partial_{\hat{p}} \underline{H}(te_1, p_1, \hat{p} - p_1 l(t))$. Therefore,

$$\begin{aligned} \partial^2_{p_1 \hat{p}} \overline{H}(te_1, 0) &= \partial^2_{p_1 \hat{p}} (\underline{H} \circ \Psi_3)(te_1, 0) \\ &= \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0) + \partial^2_{\hat{p}^2} \underline{H}(te_1, 0) l(t) \\ &= \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0) - \underline{D}(t)[\underline{D}(t)]^{-1} \partial^2_{p_1 \hat{p}} \underline{H}(te_1, 0) = 0. \end{aligned}$$

So $\overline{H}$ satisfies (3). However, $\overline{H}$ does not necessarily satisfy (4): Note that in a neighborhood of $q = 0$ we have

$$\overline{V}(q) = \partial_p \overline{H}(q, 0) = d\varphi_3^{-1}(q) \partial_p \underline{H}(q, 0) = d\varphi_3^{-1}(q) e_1 = ([1 + l'(q_1).\hat{q}]^{-1}, 0),$$

and there is no obligation for $\partial_{\hat{q}} \overline{V}_1(te_1) = \partial^2_{\hat{q} p_1} \overline{H}(te_1, 0) = -l'(t)$ to vanish. Where as before, $\overline{V}_1$ denotes for the first component of the vector $\overline{V}$.

To achieve (4), we translate $\overline{H}$ by $-k$, and then we conformally reparametrize $\overline{H} - k$ with the function $f(q) := \frac{1}{\overline{V}_1(q)}$. If we define

$$H(q,p) := f(q)\big(\overline{H}(q,p) - k\big),$$

then the Hamiltonian vector field of H takes $(te_1, 0)$, where $t \in [-\tau_1, \tau_1]$, as its orbit segment; Moreover, H satisfies both (3) and (4). □

Proof of (5). Let $\Psi_3 \circ \Psi_2 \circ \Psi_1$ be the local coordinates around $\underline{\theta}(0)$. Suppose that there exists an open neighborhood $\mathcal{U} \subset \mathbb{R}^{d+1}$ around $q = 0$ such that $\partial_p \underline{H}(q, 0) = e_1$ for all $q \in \mathcal{U}$. Set $\underline{d}_{11}(t) := \partial_{p_1^2} \underline{H}(te_1, 0)$, and $\underline{D}(t) := \partial^2_{\hat{p}^2} \underline{H}(te_1, 0)$.
Based on proof of (3), we can find $\tau_1 > 0$ such that $\big\{te_1 \mid t \in [-\tau_1, \tau_1]\big\} \subset \mathcal{U}$, and $\underline{D}(t) \in GL(d)$ for all $t \in [-\tau_1, \tau_1]$. So $\tau_2 \in (0, \tau_1)$ exists such that all matrices in the set $\{\underline{D}(t) \mid t \in [-\tau_2, \tau_2]\}$ are having the same signature. Hence, a diagonal constant matrix D with only ± 1 entries, and a smooth curve $M(t) : [-\tau_2, \tau_2] \to GL(d)$ exist so that

$$\underline{D}(t) = M(t) D M^T(t). \tag{1.4.4}$$

Define

$$\varphi_4(q) := \big(q_1, M(q_1)\hat{q}\big), \quad \Psi_4(q,p) := \big(\varphi_4(q), [d\varphi_4^{-1}(q)]^T p\big),$$

then for the Hamiltonian $\overline{H}$ defined as $\overline{H} := \underline{H} \circ \Psi_4$ we claim the following

$$\partial^2_{\hat{p}^2}\overline{H}(te_1, 0) = D. \tag{1.4.5}$$

To prove (1.4.5), we write the Taylor expansion of $\overline{H}$ with respect to p-variables around $(te_1, 0)$, where $t \in [-\tau_2, \tau_2]$:

$$\begin{aligned}\overline{H}(te_1, p) &= \underline{H} \circ \Psi_4(te_1, p) = \underline{H}\big(te_1, [d\varphi_4^{-1}(te_1)]^T p\big)\\ &= p_1 + p^T d\varphi_4^{-1}(te_1)\partial_{p^2}\underline{H}(te_1, 0)[d\varphi_4^{-1}(te_1)]^T p + O_3(p)\\ &= p_1 + p^T \begin{bmatrix} 1 & 0\\ 0 & M^{-1}(t)\end{bmatrix}\begin{bmatrix} \underline{d}_{11}(t) & 0\\ 0 & \underline{D}(t)\end{bmatrix}\begin{bmatrix} 1 & 0\\ 0 & [M^{-1}(t)]^T\end{bmatrix} p + O_3(p)\\ &= p_1 + p^T \begin{bmatrix} \underline{d}_{11}(t) & 0\\ 0 & D\end{bmatrix} p + O_3(p).\end{aligned} \tag{1.4.6}$$

Equation (1.4.6) implies what we have claimed earlier in (1.4.5).
Although $\overline{H}$ satisfies (3), we will observe that (4) is not valid for $\overline{H}$: First, note that

$$d\varphi_4(q) = \begin{bmatrix} 1 & 0\\ \dot{M}(q_1)\hat{q} & M(q_1)\end{bmatrix} \Rightarrow d\varphi_4^{-1}(q) = \begin{bmatrix} 1 & 0\\ -M^{-1}\dot{M}(q_1)\hat{q} & M^{-1}(q_1)\end{bmatrix}.$$

Therefore,

$$\begin{aligned}\partial_p\overline{H}(q, 0_n) = d\varphi_4^{-1}(q)\partial_p\underline{H}(q, 0_n) &= d\varphi_4^{-1}(q)e_1\\ &= \big(1, -M^{-1}(q_1)\dot{M}(q_1)\hat{q}\big), \quad q \in \mathcal{U}.\end{aligned} \tag{1.4.7}$$

From (1.4.7) we conclude that

$$\partial^2_{\hat{q}\hat{p}}\overline{H}(te_1, 0) = -M^{-1}(t)\dot{M}(t), \quad t \in [-\tau_2, \tau_2]. \tag{1.4.8}$$

Note that $M^{-1}(t)\dot{M}(t)$ might not vanish identically on $(te_1, 0)$ where $t \in [-\tau_2, \tau_2]$. Assertion (4) would be retained by the vertical symplectomorphism

$$\Psi_5(q, p) := \big(q, p + dg(q)\big),$$

where $g : \mathbb{R}^{d+1} \to \mathbb{R}$ satisfies

$$\partial^2_{\hat{q}^2}g(te_1) = DM^{-1}(t)\dot{M}(t), \quad dg(te_1) = 0, \quad \text{for all} \quad t \in [-\tau_2, \tau_2]. \tag{1.4.9}$$

After defining $H := \overline{H} \circ \Psi_5$, we have

$$\begin{aligned}\partial^2_{\hat{p}\hat{q}}H(te_1, 0) &= \partial^2_{\hat{p}\hat{q}}\overline{H}(te_1, 0) + \partial^2_{\hat{p}^2}\overline{H}(te_1, 0)\partial^2_{\hat{q}^2}g(te_1)\\ &= -M^{-1}(t)\dot{M}(t) + D^2M^{-1}(t)\dot{M}(t) = 0.\end{aligned} \tag{1.4.10}$$

In above, note that $D^2 = I$; Moreover, to achieve (1.4.10) we have replaced $\partial^2_{\hat{p}^2}\overline{H}(te_1, 0)$, $\partial^2_{\hat{p}\hat{q}}\overline{H}(te_1, 0)$, and $\partial^2_{\hat{q}^2}g(te_1)$ with their equivalences in equations (1.4.5), (1.4.8), and (1.4.9) respectively.
We skipped to mention that a desirable function g exists only if

$$DM^{-1}(t)\dot{M}(t) \in \mathcal{S}(d), \quad \text{for all} \quad \mathrm{t} \in [-\tau_2, \tau_2]. \tag{1.4.11}$$

The following lemma justifies the existence of a smooth curve $M(q_1) : \mathbb{R} \to GL(d)$ that satisfies both of the conditions that we have seen in (1.4.4) and (1.4.11). □

Lemma 1.4.3. *A smooth curve $M : \mathbb{R} \to GL(d)$ exists such that for all $t \in [-\tau_2, \tau_2]$ we have*

$$\underline{D}(t) = M(t)DM^T(t), \quad \textit{and} \quad DM^{-1}(t)\dot{M}(t) \in \mathcal{S}(d).$$

Proof. Let $M(t)$ be the solution of

$$\begin{cases} \dot{M}(t) = \frac{1}{2}\big(\dot{\underline{D}}(t)[M^{-1}(t)]^T D\big) \\ M(0) = M_0 \end{cases}, \quad t \in [-\tau_2, \tau_2]. \tag{1.4.12}$$

where M_0 satisfies $\underline{D}(0) = M_0 D M_0^T$. From equation (1.4.12) We have

$$DM^{-1}(t)\dot{M}(t) = \frac{1}{2}DM^{-1}(\frac{d}{dt}\underline{D})(M^{-1})^T D,$$

and the right hand side of the above equation is symmetric for all $t \in [-\tau_2, \tau_2]$.
It remains to prove that the solution of (1.4.12) satisfies the condition (1.4.11). Equation (1.4.12) immediately implies that

$$\dot{\underline{D}}(t) = 2\dot{M}(t)DM^T(t). \tag{1.4.13}$$

Because $DM^{-1}\dot{M}$ is symmetric, we have

$$\begin{aligned} &DM^{-1}\dot{M} = \dot{M}^T(M^{-1})^T D \Rightarrow M^{-1}\dot{M} = D\dot{M}^T(M^{-1})^T D \\ &\Rightarrow M^{-1}\dot{M}D = D\dot{M}^T(M^{-1})^T \\ &\Rightarrow \dot{M}DM^T = MD\dot{M}^T. \end{aligned} \tag{1.4.14}$$

From (1.4.14) and (1.4.13), we conclude that $\dot{\underline{D}}(t) = \dot{M}DM^T + MD\dot{M}^T$. Therefore,

$$\underline{D}(t) = M(t)DM^T(t) + C$$

for some constant matrix C. Since $\underline{D}(0) = M(0)DM^T(0)$, the matrix C is null. □

Proof of (6). (1) and (2) imply that $\partial_q H(te_1, 0_n) = P(t) = 0$, for all $t \in [-\delta, \delta]$. Therefore, $\partial^2_{q_1 q} H(te_1, 0) = 0$. □

Chapter 2

Perturbation theorem for non-convex Hamiltonians

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Our goal in this chapter is to prove Theorem 3 using the normal form for non-convex Hamiltonians given in Theorem 1.4.1. Theorem 3 has a crucial importance in the proof of the bumpy metric theorem that we will give in Chapter 3. See the proof of Proposition 3.2.1 where we apply Theorem 3.

If we review the statements of Theorem 3 and Theorem D, we observe that Theorem 3 is implying the other theorem in which convexity is assumed. That is because a convex Hamiltonian is iso-energetically non-degenerate at neat points. Notice the assumption "$\theta(t)$ admits a neat time t_0 such that $\theta(t_0) \notin \Gamma_H$" in the statement of Theorem 3. In the framework of convex Hamiltonians the mentioned assumption is equivalent to "$\theta(t)$ admits a neat time".

Our proof of Theorem 3 benefits from similar geometric control methods as invoked by Rifford and Ruggiero [RR12]. We begin this chapter with a review of some background about control theory on symplectic linear maps.

2.1 Background in geometric control theory

We wish to study the control problem

$$\dot{X}_w(t) = Y(t)X_w(t) + \sum_{i=1}^{k} w_i(t)B_i(t)X_w(t), \text{ for a.e. } t \in [0,\tau], \quad X_w(0) = \bar{X} \in Sp(2d). \quad (2.1.1)$$

Where in above, $Sp(2d)$ is the set of symplectic matrices of dimension $2d \times 2d$. Moreover, $Y(t)$ and $B_i(t)$, where $i \in \{1, 2, \ldots, k\}$, are belonging to the Lie algebra of $Sp(2d)$. By definition, the Lie algebra of $Sp(2d)$ is $T_I Sp(2d)$ i.e. the tangent space to $Sp(2d)$ at identity. The Lie algebra of $Sp(2d)$ is the set of matrices $M \in \mathcal{M}(2d)$ such that $\mathbb{J}M$ is symmetric, where $\mathcal{M}(2d)$ denotes for the set of all real matrices of dimension $2d \times 2d$. Recall that $\mathbb{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.
The Lie algebra of $Sp(2d)$ is identical to a space known as the *Hamiltonian matrices* which we denote by $\mathfrak{sp}(2d)$.
The control problem (2.1.1) is invariant under $Sp(2d)$. That is to say for an initial state $\bar{X} \in Sp(2d)$, the solution of (2.1.1) is a curve that remains in $Sp(2d)$.
For a given $w \in L^1\big([0,\tau];\mathbb{R}^k\big)$, and an initial state $X_w(0) = \bar{X}$, the control problem (2.1.1) admits a unique maximal solution

$$X_w(t) : I_{\bar{X},w} \subseteq [0,\tau] \to Sp(2d).$$

In above, $I_{\bar{X},w}$ which depends on both w and $\bar{X}$, is the domain of definition of the maximal solution $X_w(t)$. As a convention, whenever we write $X(t)$ without an index we refer to the solution of the homogeneous system associated to (2.1.1) i.e.

$$\dot{X}(t) = Y(t)X(t), \quad X(0) = \bar{X}. \quad (2.1.2)$$

The aim of this section is to provide sufficient conditions for *local controllability* of $X_w(\tau)$. We soon will give a precise meaning to the notion of local controllability.

Define $\mathcal{C}_{\bar{X}}$ as the set of controls $w \in L^1\big([0,\tau];\mathbb{R}^k\big)$ such that $I_{\bar{X},w} = [0,\tau]$. By definition, $\mathcal{C}_{\bar{X}} \subseteq L^1\big([0,\tau];\mathbb{R}^k\big)$ is open. Consider the *end-point mapping* $f_{\bar{X}}(w) : \mathcal{C}_{\bar{X}} \to Sp(2d)$ defined as $f_{\bar{X}}(w) := X_w(\tau)$. Whenever $df_{\bar{X}}(w)(v)$ is surjective i.e. $df_{\bar{X}}(w)\big(L^1([0,\tau];\mathbb{R}^k)\big) = T_{X_w(\tau)}Sp(2d)$, we say $f_{\bar{X}}$ is *controllable of first order* at w. We intend to convey the same meaning when we say $f_{\bar{X}}$ is *locally controllable* at w.
Note that for any given $w \in \mathcal{C}_{\bar{X}}$, differential of the end-point mapping at w is the linear operator

$$\begin{aligned} df_{\bar{X}}(w)(v) :& L^1\big([0,\tau];\mathbb{R}^k\big) \to T_{X_w(\tau)}Sp(2d) \\ & v \mapsto G_v(\tau), \end{aligned}$$

where $G_v(t)$ is the solution of the following Cauchy problem

$$\begin{cases} \dot{G}_v(t) = Y(t)G_v(t) + \sum_{i=1}^{k} v_i(t)B_i(t)X_w(t) \text{ for a.e. } t \in [0,\tau], \\ G_v(0) = 0. \end{cases} \quad (2.1.3)$$

Concerned to the problem (2.1.1), the following Lemma which has been proven in Section 2 of [RR12], and Section 2.5 of [Laz14], provides sufficient conditions for local controllability of the end-point mapping. The notation $O^\gamma(x,r)$ in the lemma indicates the Euclidean open ball of dimension γ centered at x with radius r.

Lemma 2.1.1 (Sufficient conditions for controllability of first order). *Let $f_{\bar{X}}(w) : \mathcal{C}_{\bar{X}} \to Sp(2d)$ be the end-point mapping associated to the control problem (2.1.1) where $\bar{X} \in Sp(2d)$ is given such that $0 \in \mathcal{C}_{\bar{X}}$. Assume that $Y(t), B_i(t) \in \mathfrak{sp}(2d)$ are smooth linear maps defined on $[0,\tau] \subset \mathbb{R}$. Moreover, for $j \in \mathbb{N}$, define $\{B_1^j\}, \dots, \{B_k^j\} : [0,\tau] \to \mathfrak{sp}(2d)$ as*

$$B_i^1(t) := B_i(t),$$
$$B_i^{j+1}(t) := [B_i^j(t), Y(t)] + \dot{B}_i^j,$$

where $[.,.]$ is the Lie bracket on $\mathfrak{sp}(2d)$. If there exists $\bar{t} \in [0,\tau]$ such that

$$span\big\{B_i^j(\bar{t}) | i \in \{1,2\dots,k\}, j \in \mathbb{N}\big\} = \mathfrak{sp}(2d), \tag{2.1.4}$$

then

(a) *we have $df_{\bar{X}}(0)\big(L^1([0,\tau];\mathbb{R}^k)\big) = T_{X(\tau)}Sp(2d)$ that means $f_{\bar{X}}$ is locally controllable at $w \equiv 0$.*

(b) *there exists $\mu,\nu > 0$, $p := \frac{2d(2d+1)}{2}$ smooth controls $w^1, w^2, \dots w^p : [0,\tau] \to \mathbb{R}^k$ supported on $(0,\tau)$, and a smooth map $W = (W_1, W_2, \dots, W_p) : O^{4d^2}\big(X(\tau),\mu\big) \cap Sp(2d) \to O^p(0,\nu)$, such that $W\big(X(\tau)\big) = 0$ and*

$$f_{\bar{X}}\left(\sum_{j=1}^p W_j(Z) w^j\right) = Z, \quad \text{for all} \quad Z \in O^{4d^2}\big(X(\tau),\mu\big) \cap Sp(2d).$$

We wish to note a few remarks before running into the proof of Theorem 2.2.1.
First, we introduce the *Frobenius inner product* $\langle A,B\rangle := Tr(A^T B)$ over $\mathcal{M}(2d)$. To given matrices $A, B \in \mathcal{M}(2d)$, the Frobenius inner product assigns the trace of $A^T B$.

Consider the homogeneous differential equation associated to the Cauchy problem (2.1.3) as follows

$$\begin{cases} \dot{Z}(t) = Y(t)Z(t) \\ Z(0) = I, \end{cases} \qquad t \in [0,\tau]. \tag{2.1.5}$$

Note that if Z is the solution of (2.1.5), then $\frac{d}{dt}Z^{-1}(t) = -Z^{-1}(t)Y(t)$.
Equation (2.1.6) below is obtained based on the well known relation between solutions of a non-homogeneous differential equation and its associated homogeneous equation. See Hartman [Har82], Corollary 2.1 in Chapter IV.

$$df_{\bar{X}}(0)(v) = \sum_{i=1}^k Z(\tau) \int_0^\tau v_i(t) Z^{-1}(t) B_i(t) X(t) dt, \quad v \in L^1([0,\tau];\mathbb{R}^k). \tag{2.1.6}$$

Proof of (a). Assume that the mapping $v \mapsto df_{\bar{X}}(0)(v)$ is not surjective. Then, after endowing $T_{X(\tau)}Sp(2d)$ with the Frobenius inner product, we are able to choose a non-zero matrix N in the orthogonal complement of $df_{\bar{X}}(0)\big(L^1([0,\tau];\mathbb{R}^k)\big) \subset Sp(2d)$. For such N, we have

$$\big\langle N, df_{\bar{X}}(0)(v)\big\rangle = 0, \quad \text{for all } v \in L^1([0,\tau];\mathbb{R}^k), \tag{2.1.7}$$

where $\big\langle .,.\big\rangle$ is the Frobenius inner product.
We replace $df_{\bar{X}}(0)(v)$ in equation (2.1.7) with its equivalence from equation (2.1.6), then for all

$v = (v_1, v_2, \dots, v_k) \in L^1([0,\tau], \mathbb{R}^k)$, we have

$$\sum_{i=1}^{k} \int_0^{\tau} v_i(t) \big\langle N, Z(\tau)Z^{-1}(t)B_i(t)X(t)\big\rangle dt = 0.$$

Therefore, $\big\langle N, Z(\tau)Z^{-1}(t)B_i(t)X(t)\big\rangle$ is identically equal to zero for all $i \in \{1,2,\dots,k\}$. In particular, we have

$$\big\langle N, Z(\tau)Z^{-1}(\bar{t})B_i(\bar{t})X(\bar{t})\big\rangle = 0, \quad \text{for all } i \in \{1,2,\dots,k\}. \tag{2.1.8}$$

We wish to prove that $\big\langle N, Z(\tau)Z^{-1}(t)B_i^j(t)X(t)\big\rangle$ identically vanishes for all $i \in \{1,2,\dots,k\}$ and all $j \in \mathbb{R}$. Differentiating $\big\langle N, Z(\tau)Z^{-1}(t)B_i(t)X(t)\big\rangle = 0$ with respect to t yields the following

$$\begin{aligned} &\big\langle N, -Z(\tau)Z^{-1}(t)Y(t)B_i(t)X(t)\big\rangle + \big\langle N, Z(\tau)Z^{-1}(t)\dot{B}_i(t)X(t)\big\rangle \\ +&\big\langle N, Z(\tau)Z^{-1}(t)B_i(t)YX(t)\big\rangle = 0. \end{aligned}$$

Hence, $\big\langle N, Z(\tau)Z^{-1}(t)\big(\dot{B}_i(t) + [B_i(t), Y(t)]\big)X(t)\big\rangle = \big\langle N, Z(\tau)Z^{-1}(t)B_i^2(t)X(t)\big\rangle = 0$. By induction, we conclude that

$$\big\langle N, Z(\tau)Z^{-1}(t)B_i^j(t)X(t)\big\rangle \equiv 0, \quad \text{for all } j \in \mathbb{N}, \text{ and all } i \in \{1,2,\dots,k\}. \tag{2.1.9}$$

Assume that we have

$$Z(t)Z^{-1}(\tau)NX^{-1}(t) \in \mathfrak{sp}(2d) \quad \text{for all } t \in [0,\tau], \tag{2.1.10}$$

then based on assumption (2.1.4), $Z(t)Z^{-1}(\tau)NX^{-1}(\bar{t})$ belongs to

$$span\big\{B_i^j(\bar{t}) | i \in \{1,2\dots,k\},\ j \in \mathbb{N}\big\}.$$

Therefore, if (2.1.10) is true then (2.1.9) implies that $\langle N, N\rangle = 0$ which contradicts with the assumption that N is non-zero.
We finish the proof by showing that (2.1.10) holds. That is to say $\mathbb{J}Z(t)Z^{-1}(\tau)NX^{-1}(t)$ is symmetric. From equations (2.1.5) and (2.1.2) we conclude that $Z(t) = X(t)\bar{X}^{-1}$. So we have

$$\begin{aligned} \mathbb{J}Z(t)Z^{-1}(\tau)NX^{-1}(t) &= \mathbb{J}X(t)\bar{X}^{-1}Z^{-1}(\tau)NX^{-1}(t) \\ &= \mathbb{J}X(t)\bar{X}^{-1}\bar{X}X^{-1}(\tau)NX^{-1}(t) \\ &= \mathbb{J}X(t)X^{-1}(\tau)NX^{-1}(t). \end{aligned} \tag{2.1.11}$$

If we replace $\mathbb{J}$ in the right side of (2.1.11) with $[X^{-1}(t)]^T X^T(\tau)\mathbb{J}X(\tau)X^{-1}(t)$, we have

$$\mathbb{J}Z(t)Z^{-1}(\tau)NX^{-1}(t) = [X^{-1}(t)]^T \big(X^T(\tau)\mathbb{J}N\big)X^{-1}(t). \tag{2.1.12}$$

Note that $X^{-1}(t) \in Sp(2d)$ for all $t \in [0,\tau]$, and $Sp(2d)$ forms a group. That is why $X(\tau)X^{-1}(t)$ is symplectic and we have $[X^{-1}(t)]^T X^T(\tau)\mathbb{J}X(\tau)X^{-1}(t) = \mathbb{J}$.
Recall that $N \in T_{X(\tau)}Sp(2d)$, and $T_{X(\tau)}Sp(2d) = \{X(\tau)M \mid M \in \mathfrak{sp}(2d)\}$. Therefore, there exist $M \in \mathfrak{sp}(2d)$ such that $X^{-1}(\tau)N = M$. So we conclude that

$$\mathbb{J}X^{-1}(\tau)N \in \mathcal{S}(2d).$$

Because $X^{-1}(\tau) \in Sp(2d)$, we have $\mathbb{J}X^{-1}(\tau) = X^T(\tau)\mathbb{J}$. So $X^T(\tau)\mathbb{J}N$ is symmetric as well as

$\mathbb{J}X^{-1}(\tau)N$. Since $X^T(\tau)\mathbb{J}N \in \mathcal{S}(2d)$, the right side of (2.1.12) is symmetric. □

Proof of (b). Since $\{w \in C^\infty([0,\tau];\mathbb{R}^k) \mid supp(w) \subset (0,\tau)\} \subset L^1([0,\tau];\mathbb{R}^k)$ is dense, based on part (a) there exists p smooth controls

$$w^j : [0,\tau] \to \mathbb{R}, \quad supp(w^j) \subset (0,\tau), \quad j \in \{1,2,\ldots,p\},$$

such that

$$span\{df(0)(w^j) \mid j = 1,\ldots,p\} = T_{X(\tau)}Sp(2d).$$

Choose $\gamma > 0$ such that $\sum_{j=1}^p \lambda_j w^j \in C_{\bar{X}}$, for all $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_p) \in O^p(0,\gamma)$ and all $w^j \in C^\infty([0,\tau];\mathbb{R}^k)$. Define $F : O^p(0,\gamma) \to Sp(2d)$ as

$$F(\lambda) := f_{\bar{X}}\left(\sum_{j=1}^p \lambda_j w^j\right), \quad \lambda \in O^p(0,\gamma).$$

F is smooth and $F(0) = X(\tau)$. So by the inverse function theorem, there exists $\mu,\nu > 0$ and a smooth mapping

$$W = (W_1, W_2, \ldots W_p) : O^{4d^2}\big(X(\tau),\mu\big) \cap Sp(2d) \to O^p(0,\nu), \quad W\big(X(\tau)\big) = 0,$$

such that

$$f_{\bar{X}}\left(\sum_{j=1}^p W_j(Z) w^j\right) = Z, \quad \forall Z \in O^{4d^2}\big(X(\tau),\mu\big) \cap Sp(2d).$$

□

2.2 Proof of the perturbation theorem

As it stated in Theorem 3, consider $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ as a smooth Hamiltonian. Let $\theta(t)$ be a given periodic orbit of H such that it admits a neat time t_0, and $\theta(t_0) \notin \Gamma_H$.
Assume that Σ is a transverse section to $\theta(t)$ at t_0. Without loss of generality we can assume that $t_0 = 0$, and $\theta(t)$ is in the zero energy level of H. Consider $P_u(\theta,\Sigma) : \Sigma \cap H^{-1}(0) \to \Sigma \cap H^{-1}(0)$ as the restricted Poincaré map with respect to $\theta(t)$ and the Hamiltonian vector field of $H + u$, where $u \in C^\infty_\theta(\mathbb{R}^{d+1})$. Recall Definition 2 which clarifies the notation $C^\infty_\theta(\mathbb{R}^{d+1})$.
We have defined

$$F(\theta, H+u) : C^\infty_\theta(\mathbb{R}^{d+1}) \to Sp(2d)$$

as the mapping $u \mapsto dP_u$. We wish to prove that F is weakly open.

If $F(\theta, \frac{H}{z} + u) : C^\infty_\theta(\mathbb{R}^{d+1}) \to Sp(2d)$ is weakly open so is $F(\theta, H+u) : C^\infty_\theta(\mathbb{R}^{d+1}) \to Sp(2d)$ where $z(q) : \mathbb{R}^{d+1} \to \mathbb{R}$ is a smooth non-zero function such that θ is an orbit of the Hamiltonian vector field of $\frac{H}{z}$. That is because, concerned to θ, H and $\frac{H}{z}$ are sharing the same Poincaré maps. Besides, a symplectic change of coordinates does not affect $P_u(\theta,\Sigma)$. Hence, with no loss of generality we can assume that H satisfies the assertions of Theorem 1.4.1. That is to say there exists $\delta > 0$ such that

$$H^{-1}(0) \ni \theta(t) = (te_1, 0), \quad \text{for all } t \in [-\delta,\delta];$$

Moreover, $\mathbb{J}\partial^2_{x^2}H(te_1,0)$ has the following block form for all $t\in[-\delta,\delta]$

$$\mathbb{Y}(t) := \mathbb{J}\partial^2_{x^2}H(te_1,0) = \begin{bmatrix} 0 & \mathbb{D}(t) \\ -\mathbb{K}(t) & 0 \end{bmatrix}, \tag{2.2.1}$$

where the block forms of $\mathbb{K}(t)$ and $\mathbb{D}(t)$ are as follows

$$\mathbb{K}(t) = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & K(t) & \\ 0 & & \end{bmatrix}, \; K(t) := \partial^2_{\hat{q}^2}H(te_1,0), \quad \mathbb{D}(t) = \begin{bmatrix} d_{11}(t) & \dots & 0 \\ \vdots & D & \\ 0 & & \end{bmatrix}, \; d_{11}(t) := \partial^2_{p_1^2}H(te_1,0),$$

and D is a constant diagonal matrix with only $+1$ or -1 entries on its diagonal.

2.2.1 Linearized restricted transition map

For $t\in[0,\delta]$, we define $\Lambda_t := \{q_1 = t\}$. Concerned to the orbit segment $(te_1,0)$, where $t\in[0,\delta]$, consider the one-parameter family of restricted transition maps

$$R^t : \Lambda_0 \cap H^{-1}(0) \to \Lambda_t \cap H^{-1}(0).$$

For a given $t\in[0,\delta]$, R^t is defined in a neighborhood of 0 and the image of $x\in\Lambda_0\cap H^{-1}(0)$ under the map R^t is where the Hamiltonian flow at the point x encounters the section Λ_t.
Note that because of the properties of the normal form given in Theorem 1.4.1 we have $dH(te_1,0) = \begin{bmatrix} 0 \\ e_1 \end{bmatrix}$ which implies the following

$$\langle dH(te_1,0), x\rangle = p_1. \tag{2.2.2}$$

Equation (2.2.2) in above implies that $\{p_1 = 0\}$ is the tangent space to the zero energy level of H along the orbit segment $(te_1,0)$, where $t\in[0,\delta]$. Therefore, the differential of R^t with respect to x can be viewed as a map from $\{q_1=0,\ p_1=0\}$ to itself:

$$dR^t : \{q_1=0,\ p_1=0\} \to \{q_1=0,\ p_1=0\}.$$

We consider $R^t_u : \Lambda_0\cap(H+u)^{-1}(0) \to \Lambda_t\cap(H+u)^{-1}(0)$ as a one-parameter family of restricted transition maps with respect to $\theta(t)$ and the Hamiltonian vector field of $H+u$, where $u\in C^\infty_\theta(\mathbb{R}^{d+1})$. Equation (2.2.2) is invariant under adding an admissible potential to H. That is to say if $u\in C^\infty_\theta(\mathbb{R}^{d+1})$, then $\langle d(H+u)(te_1,0), x\rangle = p_1$. Moreover, for each admissible potential u, we have $(H+u)(te_1,0)=0$. So $\{p_1=0\}$ is the tangent space to $(H+u)^{-1}(0)$ along the orbit segment $(te_1,0)$, where $t\in[0,\delta]$. In conclusion, likewise dR^t, the differential with respect to x of the perturbed restricted transition map, namely dR^t_u, is a map from $\{q_1=0,\ p_1=0\}$ to itself:

$$dR^t_u : \{q_1=0,\ p_1=0\} \to \{q_1=0,\ p_1=0\}.$$

We indicate the differentials at $x=0$ of dR^t and dR^t_u by $L(t)$ and $L_u(t)$ respectively. Both $L(t)$ and $L_u(t)$ are valued in $Sp(2d)$.

After recalling the definition of $\mathbb{Y}(t)$ from equation (2.2.1), consider the differential equation

$$\dot{W}(t) = \mathbb{Y}(t)W(t), \quad W(0) = I, \quad t\in[0,\delta]. \tag{2.2.3}$$

which is known as the *Jacobi equation* (see [Con10]). In the following computations, using the definition of the Hamiltonian flow of H, we show that $W(t)$ is equal to the differential of the

Hamiltonian flow at $x = 0$. That means $W(t) = \partial_x \phi^t(0)$.

$$\partial_t \partial_x \phi^t(x) = \partial_x \partial_t \phi^t(x) = \partial_x \mathbb{J} dH\big(\phi^t(x)\big) \Rightarrow \partial_t \partial_x \phi^t(0) = \mathbb{Y}(t) \partial_x \phi^t(0).$$

In what follows, we are considering the linearized Hamiltonian system of H along $(te_1, 0)$, where $t \in [0, \delta]$, and the initial point belongs to $\{q_1 = 0,\ p_1 = 0\}$

$$\dot{x}(t) = \mathbb{Y}(t) x(t), \quad x(0) = x^0 \in \{q_1 = 0,\ p_1 = 0\}. \tag{2.2.4}$$

If $W(t)$ solves the differential equation (2.2.3), then the mapping $x^0 \mapsto W(t)x^0$ is the flow of the system (2.2.4).

As it is usual in this thesis we use the notations $x = (q, p)$, $x_1 = (q_1, p_1)$, and $\hat{x} = (\hat{q}, \hat{p})$ for symplectic coordinates where $q = (q_1, \hat{q}) \in \mathbb{R} \times \mathbb{R}^d$, and $p = (p_1, \hat{p}) \in \mathbb{R}^* \times (\mathbb{R}^d)^*$. Because of the intrinsic properties of dR^t, it is convenient to decompose coordinates as $x = (x_1, \hat{x}) = (q_1, p_1, \hat{x})$ in the proof of Lemma 2.2.1 below. By writing $x = (q_1, p_1, \hat{x})$ we do not mean to change the label of the dual symplectic coordinates (q, p), but we aim to study the impact of the restricted transition maps on x_1-coordinates and $\hat{x}$-coordinates separately.
In the statements of Lemma 2.2.1 and Corollary 2.2.2 below, we define $Y(t) := \mathbb{J}\partial^2_{\hat{x}^2} H(te_1, 0)$, and $Y_u(t) := \mathbb{J}\partial^2_{\hat{x}^2}(H + u)(te_1, 0)$ where $u \in C^\infty_\theta(\mathbb{R}^{d+1})$. Note that since we are working in the coordinates of Theorem 1.4.1, the block forms of $Y(t)$ and $Y_u(t)$ are as follows

$$Y(t) = \begin{bmatrix} 0 & D \\ -K(t) & 0 \end{bmatrix}, \quad Y_u(t) = \begin{bmatrix} 0 & D \\ -K_u(t) & 0 \end{bmatrix},$$

where

$$K(t) := \partial^2_{\hat{q}^2} H(te_1, 0), \quad K_u(t) := K(t) + \partial^2_{\hat{q}^2} u(te_1),$$

and D is a diagonal matrix which has only $+1$ or -1 entries on its diagonal.

Lemma 2.2.1. *Assume that a smooth Hamiltonian $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ takes $(te_1, 0) \in H^{-1}(0)$ as an orbit segment where $t \in [0, \delta]$ for some $\delta > 0$, and H satisfies the assertions (3) to (5) of Theorem 1.4.1 on this orbit segment. Then, $L(t) := dR^t(0)$ solves the differential equation*

$$\dot{L}(t) = Y(t)L(t),$$

where $Y(t) := \mathbb{J}\partial^2_{\hat{x}^2} H(te_1, 0)$, and $R^t : \{q_1 = 0\} \cap H^{-1}(0) \to \{q_1 = t\} \cap H^{-1}(0)$ is the one-parameter family of restricted transition maps associated to the segment $(te_1, 0)$, $t \in [0, \delta]$.

Proof. The block form of $\mathbb{Y}(t)$ allows us to rewrite the system (2.2.4) as two uncoupled systems

$$(1) \begin{cases} \dot{q}_1(t) = d_{11}(t) p_1(t) \\ \dot{p}_1(t) = 0 \end{cases}, \quad (2)\ \frac{d}{dt}\hat{x}(t) = Y(t)\hat{x}, \quad x(0) = x^0 \in \{q_1 = 0,\ p_1 = 0\}. \tag{2.2.5}$$

We let $L(t)$ be the solution of the following differential equation

$$\dot{L}(t) = Y(t)L(t), \quad L(0) = I,$$

then we show that $L(t)$ is equivalent to $dR^t(0)$.
The map $\hat{x}^0 \mapsto L(t)\hat{x}^0$ is the flow associated with system (2) in (2.2.5) above. At the other hand, recall that for $W(t) = \partial_x \phi^t(0)$, the map $x^0 \mapsto W(t)x^0$ is the flow subjected to the system (2.2.4).

Therefore, because systems (1) and (2) in (2.2.5) are uncoupled we can write

$$W(t)\begin{bmatrix}0\\0\\\hat{x}^0\end{bmatrix} = \begin{bmatrix}0\\0\\L(t)\hat{x}^0\end{bmatrix}.$$

We write the linear approximation of the Hamiltonian flow of H around $0 \in T^*\mathbb{R}^{d+1}$

$$\phi^t(q_1, p_1, \hat{x}) = \phi^t(0) + W(t)\begin{bmatrix}q_1\\p_1\\\hat{x}\end{bmatrix} + O_2(x) = \begin{bmatrix}t\\0\\0\end{bmatrix} + W(t)\begin{bmatrix}q_1\\p_1\\\hat{x}\end{bmatrix} + O_2(x).$$

The restriction of the above approximation to $\{q_1 = 0,\ p_1 = 0\}$ is

$$\phi^t(0,0,\hat{x}) = \begin{bmatrix}t\\0\\0\end{bmatrix} + W(t)\begin{bmatrix}0\\0\\\hat{x}\end{bmatrix} + O_2(\hat{x}) = \begin{bmatrix}t\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\L(t)\hat{x}\end{bmatrix} + O_2(\hat{x}). \tag{2.2.6}$$

The right side of (2.2.6) is nothing but the linear approximation of R^t around 0. We conclude that $\partial_{\hat{x}}\phi^t(0) = dR^t(0)$, but from (2.2.6), we have $\partial_{\hat{x}}\phi^t(0) = L(t)$. In conclusion, $L(t) = dR^t(0)$. □

Corollary 2.2.2. *Assume that $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ is smooth and for some $\delta > 0$, $\theta(t) = (te_1, 0) \in H^{-1}(0)$ where $t \in [0,\delta]$, is an orbit segment of the Hamiltonian vector field of H. Moreover, suppose that H satisfies the assertions (3) to (5) of Theorem 1.4.1 on $\theta(t)$ for all $t \in [0,\delta]$. For $u \in C^\infty_\theta(\mathbb{R}^{d+1})$, suppose that $R^t_u : \{q_1 = 0\} \cap (H+u)^{-1}(0) \to \{q_1 = t\} \cap (H+u)^{-1}(0)$ is the one-parameter family of restricted transition maps with respect to the orbit segment $\theta(t)$, $t \in [0,\delta]$, and the Hamiltonian vector field of $H+u$. Then $L_u(t) := dR^t_u(0)$ solves the differential equation*

$$\dot{L}_u(t) = Y_u(t)L_u(t),$$

where $Y_u(t) := \mathbb{J}\partial^2_{\hat{x}^2}(H+u)(te_1, 0)$.

To conclude Corollary 2.2.2 from Lemma 2.2.1, it is enough to show that if a smooth Hamiltonian $H : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ takes $\theta(t) = (te_1, 0)$, where $t \in [0,\delta]$, as an orbit segment and it satisfies the assertions of Theorem 1.4.1 on this segment, so does $H + u$ where $u \in C^\infty_\theta(\mathbb{R}^{d+1})$. If $u \in C^\infty_\theta(\mathbb{R}^{d+1})$, then $(te_1, 0)$ is an orbit segment in the zero energy level of $H+u$ as well. Furthermore, we have $\partial^2_{q_1 q}u(te_1) = 0$ for all $t \in [0,\delta]$. Therefore, in comparison between $\mathbb{Y}_u(t)$ and $\mathbb{Y}(t)$, the only difference appears in the minor blocks $K_u(t) = \partial^2_{\hat{q}^2}(H+u)(te_1, 0)$ and $K(t) = \partial^2_{\hat{q}^2}H(te_1, 0)$. That means $H + u$, where $u \in C^\infty_\theta(\mathbb{R}^{d+1})$, satisfies assertions of Theorem 1.4.1 on $\theta(t)$ for all $t \in [0,\delta]$.

2.2.2 Perturbed transition maps from the viewpoint of control theory

Corollary 2.2.2 declares that $dR^t_u(0) =: L_u(t)$ is the solution of the following differential equation

$$\dot{L}_u(t) = Y_u(t)L_u(t), \quad L_u(0) = I, \quad u \in C^\infty_\theta(\mathbb{R}^{d+1}), \tag{2.2.7}$$

where $Y_u(t) := \mathbb{J}\partial^2_{\hat{x}^2}(H+u)(te_1,0)$. Equation (2.2.7) in above can be viewed as the control problem

$$\dot{X}_w(t) = Y(t)X_w(t) + \sum_{\substack{i,j=1\\ i\le j}}^{d} w_{ij}(t)\begin{bmatrix}0 & 0\\ E(ij) & 0\end{bmatrix} X_w(t), \quad t\in[0,\delta], \quad X_w(0)=I, \tag{2.2.8}$$

where w_{ij} represents the coordinates of the control $w \in C^\infty([0,\delta],\mathcal{S}(d))$. Moreover, $E(ij)$ is the symmetric $d\times d$ binary matrix that its only non-zero components are placed at ij and ji entries. It is easy to see that $\{E(ij) \mid i,j\in\{1,2,\dots,d\}\}$ is a basis for $\mathcal{S}(d)$ which denotes for the set of symmetric matrices of dimension $d\times d$.

In the proof of the Proposition 2.2.3 below, using Lemma 2.1.1 we will show that there exists an open dense subset $U \subset C^\infty_\theta(\mathbb{R}^{d+1})$ such that for each $\bar{u}\in U$ the end-point mapping $f: C^\infty([0,\delta];\mathcal{S}(d)) \to Sp(2d)$ associated to the control problem

$$\dot{\mathcal{X}}_w(t) = Y_{\bar{u}}(t)\mathcal{X}_w(t) + \sum_{\substack{i,j=1\\ i\le j}}^{d} w_{ij}(t)\begin{bmatrix}0 & 0\\ E(ij) & 0\end{bmatrix} \mathcal{X}_w(t), \quad t\in[0,\delta], \quad \mathcal{X}_w(0)=I, \tag{2.2.9}$$

is controllable of first order at $w\equiv 0$. That is equivalent to say that the differential of f at $w\equiv 0$ is onto i.e.

$$df(0)\big(C^\infty\big([0,\delta];\mathcal{S}(d)\big)\big) = T_{\mathcal{X}(\delta)}Sp(2d), \tag{2.2.10}$$

where in the right hand side of (2.2.10), $\mathcal{X}(t)$ denotes for the solution of the following homogeneous equation

$$\dot{\mathcal{X}}(t) = Y_{\bar{u}}(t)\mathcal{X}(t), \quad \mathcal{X}(0)=I.$$

Recall that the end-point mapping with respect to the control problem (2.2.9) is defined as $w\mapsto \mathcal{X}_w(\delta)$, where $\mathcal{X}_w(t)$ is the solution of (2.2.9).

Proposition 2.2.3. *There exists an open dense subset $U\subset C^\infty_\theta(\mathbb{R}^{d+1})$ such that for a given $\bar{u}\in U$, the end-point mapping $w\mapsto \mathcal{X}_w(\delta)$ associated to the control problem*

$$\dot{\mathcal{X}}_w(t) = Y_{\bar{u}}(t)\mathcal{X}_w(t) + \sum_{\substack{i,j=1\\ i\le j}}^{d} w_{ij}(t)\begin{bmatrix}0 & 0\\ E(ij) & 0\end{bmatrix} \mathcal{X}_w(t), \quad t\in[0,\delta], \quad \mathcal{X}_w(0)=I,$$

is locally controllable at $w\equiv 0$.

Proof. Let $\bar{t}\in[0,\delta]$ be given. Based on Lemma 2.2.1, it is enough to prove that

$$span\{B^1_{ij}(\bar{t}), B^2_{ij}(\bar{t}), B^3_{ij}(\bar{t}), B^4_{ij}(\bar{t}) \mid i,j\in\{1,2,\dots,d\}\} = \mathfrak{sp}(2d), \tag{2.2.11}$$

where

$$B^1_{ij}(t) = \begin{bmatrix}0 & 0\\ E(ij) & 0\end{bmatrix},$$
$$B^r_{ij}(t) = [B^{r-1}_{ij}(t), Y_{\bar{u}}(t)], \quad r=2,3,4.$$

We have

$$B^2_{ij}(\bar t) = \begin{bmatrix} -DE(ij) & 0 \\ 0 & E(ij)D \end{bmatrix},$$

$$B^3_{ij}(\bar t) = \begin{bmatrix} 0 & -2DE(ij)D \\ -E(ij)DK_{\bar u}(\bar t) - K_{\bar u}(\bar t)DE(ij) & 0 \end{bmatrix},$$

$$B^4_{ij}(\bar t) = \begin{bmatrix} 3DE(ij)DK_{\bar u}(\bar t) + DK_{\bar u}(\bar t)DE(ij) & 0 \\ -E(ij)DK'_{\bar u}(\bar t) - K'_{\bar u}(\bar t)DE(ij) & -E(ij)DK_{\bar u}(\bar t)D - 3K_{\bar u}(\bar t)DE(ij)D \end{bmatrix}.$$

Note that $\tilde B^4_{ij}(\bar t)$ defined as follows is in the span of $B^1_{ij}(\bar t)$ and $B^4_{ij}(\bar t)$

$$\tilde B^4_{ij}(\bar t) = \begin{bmatrix} = 3DE(ij)DK_{\bar u}(\bar t) + DK_{\bar u}(\bar t)DE(ij) & 0 \\ 0 & -E(ij)DK_{\bar u}(\bar t)D - 3K_{\bar u}(\bar t)DE(ij)D \end{bmatrix}.$$

Since $span\big\{B^1_{ij}(\bar t), B^2_{ij}(\bar t), B^3_{ij}(\bar t), \tilde B^4_{ij}(\bar t) \mid i,j \in \{1,2,\dots,d\}\big\} \subseteq \mathfrak{sp}(2d)$, in order to prove (2.2.11) we just need to show that

$$dim\big(span\big\{B^1_{ij}(\bar t), B^2_{ij}(\bar t), B^3_{ij}(\bar t), \tilde B^4_{ij}(\bar t) \mid i,j \in \{1,2,\dots,d\}\big\}\big) = \frac{2d(2d+1)}{2}.$$

As $DE(ij)D$ is a basis for $\mathcal{S}(d)$, the dimension of $span\big\{B^1_{ij}(\bar t), B^3_{ij}(\bar t) \mid i,j \in \{1,2,\dots,d\}\big\}$ is equal to $d(d+1)$. Therefore, because we have

$$span\big\{B^1_{ij}(\bar t), B^3_{ij}(\bar t) | i,j \in \{1,2,\dots,d\}\big\} \cap span\big\{B^2_{ij}(\bar t), B^4_{ij}(\bar t) \mid i,j \in \{1,2,\dots,d\}\big\} = 0,$$

equation (2.2.11) holds whenever

$$span\big\{B^2_{ij}(\bar t), \tilde B^4_{ij}(\bar t) | i,j \in \{1,2,\dots,d\}\big\} = \frac{2d(2d+1)}{2} - d(d+1) = d^2. \tag{2.2.12}$$

Define $\mathcal{G} := \big\{ -E(ij), 3E(ij)DK_{\bar u}(\bar t) + K_{\bar u}(\bar t)DE(ij) \mid i,j \in \{1,2,\dots,d\}\big\}$, then (2.2.12) is true if $\mathcal{G}$ spans $\mathcal{M}(d)$ which denotes for the space of real $d \times d$ matrices.
Note that $\mathcal{M}(d) = \mathcal{S}(d) \oplus \mathcal{S}^-(d)$ —where $\mathcal{S}^-(d)$ denotes for the anti-symmetric $d \times d$ matrices— and $\mathcal{G}$ already includes a basis for symmetric matrices. Define

$$Z(ij) := E(ij)DK_{\bar u}(\bar t) - K_{\bar u}(\bar t)DE(ij), \quad i<j, \quad i,j \in \{1,2,\dots,d\},$$

which is the skew-symmetric part of $3E(ij)DK_{\bar u}(\bar t) + K_{\bar u}(\bar t)DE(ij)$ in the decomposition alike $M = \frac{1}{2}(M + M^T) - \frac{1}{2}(M - M^T)$, for $M \in \mathcal{M}(d)$. Consider the linear function

$$\Omega : \mathcal{S}(d) \to \big(\mathcal{S}^-(d)\big)^{\frac{d(d-1)}{2}}$$

that the coordinates of its image are defined as

$$\Omega_{ij}(S) := \big(E(ij)DS - SDE(ij)\big) \quad i,j \in \{1,2,\dots,d\}, \quad i<j.$$

For S_0 such that $DS_0 = diag(1,2,\dots,d)$, it is easy to check that $\det \Omega(S_0) \neq 0$; In fact, $\Omega_{ij}(S_0)$ are equal to $(j-i)E(ij)$ which are linearly independent. Since $\det \Omega(S_0) \neq 0$, we conclude that the determinant of Ω is not identically equal to zero. Therefore, the set R defined as follows

$$R := \big\{S \in \mathcal{S}(d) \mid \text{coordinates of } \Omega(S) \text{ are forming a basis for } \mathcal{S}^-(d)\big\},$$

is the complement of an Algebraic set. Hence, R is an open dense subset of $\mathcal{S}(d)$.

We define

$$U := \{u \in C^\infty_\theta(\mathbb{R}^{d+1}) \mid K_u(\bar{t}) \in R\}.$$

Note that for each $\bar{u} \in U$, equation (2.2.12) is true. So to finish the proof, it remains to show that U is an open and dense subset of $C^\infty_\theta(\mathbb{R}^{d+1})$. Consider the map $g(u) : C^\infty_\theta(\mathbb{R}^{d+1}) \to \mathcal{S}(d)$ defined as $g(u) := K_u(t)$. Because $g(u)$ is open and continuous, and $R \subset \mathcal{S}(d)$ is open and dense, then $g^{-1}(R) = U$ is an open and dense subset of $C^\infty_\theta(\mathbb{R}^{d+1})$. □

2.2.3 Proof of Theorem 3

Proposition 2.2.3 and part (b) of Lemma 2.2.1 imply that there exists an open dense subset $U \subset C^\infty_\theta(\mathbb{R}^{d+1})$ and a finite dimensional subspace $F \subset C^\infty\big([0,\delta];\mathcal{S}(d)\big)$ such that for a given $\bar{u} \in U$, the map

$$F \ni w \mapsto \mathcal{X}_w(\delta)$$

is a C^1 submersion near $w \equiv 0$. Where $\mathcal{X}_w(t)$ is the solution of the control problem (2.2.9). Suppose that T is the minimum period of $\theta(t)$. We define

$$\mathcal{Y} := \{u \in C^\infty_\theta(\mathbb{R}^{d+1}) \mid \ d^2u\big(\pi \circ \theta(]\delta, T+\delta[)\big) = 0\}.$$

There exists a finite dimensional subspace $E \subset \mathcal{Y}$ such that the map $h : E \to F$ defined as $h(u) := \partial^2_{\hat{q}^2} u(te_1)$ is a linear isomorphism where $t \in [0,\delta]$. In consequence, the map

$$E \ni u \mapsto L_{\bar{u}+u}(\delta)$$

is a C^1 submersion near $u \equiv 0$.

We wish to prove that the map $F(\theta, H+u) : C^\infty_\theta(\mathbb{R}^{d+1}) \to Sp(2d)$ defined as $u \mapsto dP_u$ is weakly open.

For a given $u_0 \in C^\infty_\theta(\mathbb{R}^{d+1})$, there exist linear symplectic maps

$$V_{u_0} : T\Sigma \cap TH^{-1}(0) \to T\Lambda_0 \cap TH^{-1}(0), \quad Q_{u_0} : T\Lambda_\delta \cap TH^{-1}(0) \to T\Sigma \cap TH^{-1}(0),$$

such that $dP_{u_0} = V_{u_0} L_{u_0}(\delta) Q_{u_0}$.

Let $O \subset C^\infty_\theta(\mathbb{R}^{d+1})$ be a given open subset. Since $U \subset C^\infty_\theta(\mathbb{R}^{d+1})$ is open and dense, $O \cap U$ is a non-empty open subset of $C^\infty_\theta(\mathbb{R}^{d+1})$. We choose $\bar{u} \in O \cap U$. There exists a finite dimensional subspace $E \subset \mathcal{Y}$ such that the map $E \ni u \mapsto L_{\bar{u}+u}(\delta)$ is a C^1 submersion near $u \equiv 0$. Therefore, $dP_{\bar{u}+u} = V_{\bar{u}} L_{\bar{u}+u} Q_{\bar{u}}$ is a C^1 submersion near $u \equiv 0$.

2.3 Two remarks on assumptions of Theorem 3

In this section we wish to show that "θ admits a neat time t_0 such that $\theta(t_0) \notin \Gamma_H$" in the statement of Theorem 3 is a necessary assumption.

First we show that admitting a neat time is a necessary assumption. Consider the smooth Hamiltonian

$$H(q,p) = g_q(p,p) + u(q), \tag{2.3.1}$$

where g is a Riemannian metric and $u(q)$ is a potential. Assume that θ is a periodic symmetric orbit of H. Recall that an orbit is called *symmetric* if it does not admit any neat time. As we mentioned in the introduction of this thesis, Kozlov [Koz76] proves the existence of a periodic symmetric orbit for such a Hamiltonian H of the form (2.3.1).

A map $\Re : T^*M \to T^*M$ is a *reversing involution* of the phase space if for a given $\alpha_q \in T^*_qM$ it satisfies $\Re(\alpha_q) = -\alpha_q$. A Hamiltonian $H(q,p) : T^*\mathbb{R}^{d+1} \to \mathbb{R}$ is called *reversible* whenever $H = H \circ \Re$, where $\Re$ is a reversing involution on T^*M. Note that for a reversible Hamiltonian H, if $\big(Q(t), P(t)\big)$ is an orbit of the Hamiltonian vector field of H, then $\big(Q(-t), -P(-t)\big)$ is also an orbit.
Let ϕ^t be the Hamiltonian flow of a reversible Hamiltonian H, then for each $x \in T^*M$, the flow satisfies the following

$$\big(\phi^t \circ \Re \circ \phi^t\big)(x) = \Re(x), \tag{2.3.2}$$

where $\Re$ is a reversing involution on T^*M.

The Hamiltonian H given in equation (2.3.1) is reversible. That is simply because $g_q(p,p)$ is quadratic with respect to p-variable.

To have a geometric intuition of periodic symmetric orbits of a reversible Hamiltonian H, note that if $\theta(t) = \big(Q(t), P(t)\big)$ is a periodic symmetric orbit of H with minimal period T, then $\theta(t)$ where $t \in [0,T]$, meets the zero section exactly twice. These intersection points are those points where the orbit $\theta(t)$ has zero velocity i.e. points at which $\dot{Q}(t)$ vanishes. Except from these two points, $\theta(t)$ where $t \in [0,T]$, meets each vertical fibration exactly twice.

Definition 2.3.1. *For $d \geq 1$, $N \in Sp(2d)$ is a reversible symplectic matrix if it satisfies*

$$NRN = R, \quad \textit{where } R = \begin{bmatrix} I_d & 0 \\ 0 & -I_d \end{bmatrix}_{2d\times 2d}.$$

Proposition 2.3.2. *Assume that $H : T^*M \to T^*M$ is a smooth reversible Hamiltonian defined on cotangent bundle of a smooth $(d+1)$-dimensional manifold M where $d \geq 1$. Suppose that $\theta(t) \in H^{-1}(k)$ is a symmetric periodic orbit of Hamiltonian vector field of H. Let*

$$P : \{p_1 = 0\} \cap H^{-1}(k) \to \{p_1 = 0\} \cap H^{-1}(k)$$

be the restricted Poincaré map with respect to $\theta(t)$, then the differential of P is a reversible symplectic matrix.

If $\theta(t)$ is a periodic symmetric orbit of $H : T^*M \to \mathbb{R}$, then $\theta(t)$ is a periodic symmetric orbit of $H + u$ where $u \in C^\infty_\theta(M)$. The set of all reversible symplectic matrices is a submanifold of $Sp(2d)$ with positive codimension. Therefore, assuming Proposition 2.3.2, the image of the map $F(\theta, H+u)$ defined as

$$C^\infty_\theta(M) \ni u \mapsto dP_u,$$

for a periodic symmetric orbit θ of a reversible Hamiltonian H, has no interior in $Sp(2d)$. Hence, Theorem 3 does not hold for such orbits.

Proof of Proposition 2.3.2. Let ϕ^t be the Hamiltonian flow of H, and $\tau(x) : \{p_1 = 0\} \to \mathbb{R}^+$ be the first return time to the section $\{p_1 = 0\}$. We wish to solve the equation

$$\big(\phi^{s(x)} \circ \Re \circ \phi^{\tau(x)}\big)(x) = \Re(x), \quad x \in \{p_1 = 0\}, \tag{2.3.3}$$

where $s(x) : x \in \{p_1 = 0\} \to \mathbb{R}^+$ is unknown and $\Re(x) : T^*M \to T^*M$ is a reversing involution. Consider the mapping $p_1(x) : T^*M \to \mathbb{R}$ that gives the p_1-coordinate of a given point $x \in T^*M$. For each $x \in \{p_1 = 0\}$, we have

$$\big(p_1 \circ \Re \circ \phi^{\tau(x)}\big)(x) = 0, \quad x \in \{p_1 = 0\}. \tag{2.3.4}$$

Therefore, from equations (2.3.2) and (2.3.4) we conclude that $\tau(x)$ solves the equation (2.3.3). Hence, for $x \in \{p_1 = 0\}$ we have $\big(\phi^{\tau(x)} \circ \Re \circ \phi^{\tau(x)}\big)(x) = \Re(x)$ which implies that

$$P \circ \Re \circ P = \Re. \tag{2.3.5}$$

Differentiating (2.3.5) gives

$$(dP)R(dP) = R, \quad \text{where } R = \begin{bmatrix} I_d & 0 \\ 0 & -I_d \end{bmatrix}_{2d \times 2d}.$$

We proved that dP is a reversible symplectic matrix. □

To complete this section, we will study an example of a Hamiltonian vector field that all its orbits are periodic and they are admitting neat points only, but they are all included in $\mathit{\Gamma}_H$. Let $M = S^1 \times \mathbb{R}^d$, where S^1 is the unit circle. We define $H(q,p) : T^*M \to \mathbb{R}$ as $H(q,p) = p_1$. The Hamiltonian system associated to H is as follows

$$\dot{q}_1 = 1, \ \frac{d}{dt}\hat{q} = 0, \quad \dot{p} = 0.$$

Therefore, all orbits are periodic orbits that are consisting of neat points only and they are all included in $\mathit{\Gamma}_H$. Note that the Poincaré map of a given orbit of the above system is fixing the q-coordinates, so the first block line of the linearized Poincaré map of a given orbit is $[I, 0]$, and this block line is invariant under admissible perturbations. Such matrices with first block line $[I, 0]$ have no interior in $Sp(2d)$. So the map $F(\theta, H + u)$ for an orbit θ of the Hamiltonian H is not weakly open.

Chapter 3

Bumpy metric theorem

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The aim of this chapter is to prove Theorem 4 and Theorem 5. Then, as we have explained in the introduction of this thesis, Theorem 6 would be an immediate consequence which in the context of this thesis we name it as bumpy metric theorem.

3.1 Background

3.1.1 Preliminaries on projection map

In this section we study the projection map $\pi_Y : X \times Y \to Y$ where X and Y are topological spaces. Our goal is to show that whenever X is a countable union of compact subsets, the image of every F_σ subset under π_Y is F_σ; See Lemma 3.1.3 below. The mentioned fact would be a consequence of the so-celled *tube lemma* which can be found in standard text books of general topology, look at Munkres [Mun00] Lemma 26.8 for example.
Assume that X and Y are two topological spaces. Consider the product space $X \times Y$ with the product topology. For a singleton $\{y_0\} \subset Y$, the subset $X \times \{y_0\} \subset X \times Y$ is called a *slice*. Given a slice $X \times \{y_0\} \subset X \times Y$, if $U_{y_0} \subset Y$ is an open set containing y_0, then $X \times U_{y_0}$ is called a *tube* around the slice $X \times \{y_0\}$.

Lemma 3.1.1 (Tube lemma)**.** *Let X and Y be two topological spaces and assume that X is compact. Consider the product space $X \times Y$, and suppose that $\mathcal{O} \subset X \times Y$ is an open subset containing the slice $X \times \{y_0\} \subset X \times Y$. There exists an open subset U_{y_0} such that $y_0 \in U_{y_0}$, and the tube $X \times U_{y_0}$ is contained in $\mathcal{O}$.*

Proof. Let $\bigcup_\alpha(\mathcal{T}_\alpha \times \mathcal{E}_\alpha)$ be an open covering of $X \times \{y_0\}$ such that $\bigcup_\alpha(\mathcal{T}_\alpha \times \mathcal{E}_\alpha) \subset \mathcal{O}$ and $\mathcal{T}_\alpha \times \mathcal{E}_\alpha$ are basis elements of the product topology concerned to the product space $X \times Y$. Because $X \times \{y_0\}$ is homeomorphic to X and X is a compact space, we conclude that the slice $X \times \{y_0\} \subset X \times Y$ is compact. Therefore, $X \times \{y_0\}$ can be covered by finite elements among $\mathcal{T}_\alpha \times \mathcal{E}_\alpha$, namely $\mathcal{T}_i \times \mathcal{E}_i$, where $i \in \{1, 2, \dots n\}$. After eliminating those elements $\mathcal{T}_j \times \mathcal{E}_j$ such that $y_0 \notin \mathcal{E}_j$, we still have a finite covering $\bigcup_i(\mathcal{T}_i \times \mathcal{E}_i)$, where $i \in \{1, 2 \dots, m\}$ and $m \leq n$. We define $U_{y_0} := \bigcap_i \mathcal{T}_i$, then $X \times U_{y_0}$ is the desired tube. □

Lemma 3.1.2. *Assume that X and Y are two topological spaces and X is compact. Consider $\pi_Y : X \times Y \to Y$ as the projection map $(x, y) \mapsto y$. Let $\mathcal{C}$ be a closed subset of $X \times Y$, then $\pi_Y(\mathcal{C}) \subset Y$ is closed.*

Proof. Suppose that y_0 is in the complement of $\pi_Y(\mathcal{C}) \subset Y$. Then, $\mathcal{C}^c$ contains the slice $X \times \{y_0\}$. Note that $\mathcal{C}^c$ is an open subset of $X \times Y$, and by assumption X is compact. So based on the tube lemma there exists an open set $U_{y_0} \subset Y$ such that $y_0 \in U_{y_0}$ and $X \times U_{y_0}$ is contained in $\mathcal{C}^c$. Therefore, we have $\pi_Y(X \times U_{y_0}) \subset \pi_Y(\mathcal{C}^c)$ which implies that U_{y_0} is contained in the complement of $\pi(\mathcal{C}) \subset Y$. We proved that a given point y_0 in the complement of $\pi_Y(\mathcal{C})$ is an interior point, so $\pi_Y(\mathcal{C}) \subset Y$ is closed. □

Once again, consider the projection map $\pi_Y : X \times Y \to Y$ where X and Y are topological spaces and X is compact. Moreover, assume that $\mathcal{F} \subset X \times Y$ is a given F_σ subset. Then, Lemma 3.1.1 implies that $\pi_Y(\mathcal{F}) \subset Y$ is F_σ: Since $\mathcal{F} \subset X \times Y \to Y$ is F_σ we can write $\mathcal{F} = \bigcup_i \mathcal{C}_i$ as a countable union of closed sets $C_i \subset X \times Y$ where $i \in \mathbb{N}$. Therefore, we have $\pi(\mathcal{F}) = \pi(\bigcup_i \mathcal{C}_i) = \bigcup_i \pi(\mathcal{C}_i)$. Because of Lemma 3.1.1, $\pi(\mathcal{C}_i) \subset Y$ is closed for all $i \in \mathbb{N}$, so we have just wrote $\pi(\mathcal{F})$ as a countable union of closed subsets which means $\pi(\mathcal{F})$ is an F_σ subset of Y.

Now consider the same projection map $\pi_Y : X \times Y \to Y$, but this time assume that X is a countable union of compact sets. For such a map, using Lemma 3.1.1 we can deduce that the image of each closed subset is an F_σ subset.

Lemma 3.1.3. *Assume that X and Y are two topological spaces and X is a countable union of compact sets. Consider $\pi_Y : X \times Y \to Y$ as the projection map $(x, y) \mapsto y$. If $\mathcal{C} \subset X \times Y$ is a given closed subset, then $\pi(\mathcal{C}) \subset Y$ is F_σ.*

Proof. By assumption, we can write X as a countable union of compact subsets $X_i \subset X$, where $i \in \mathbb{N}$,

$$X = \bigcup_{i \in \mathbb{N}} X_i.$$

For each $i \in \mathbb{N}$, let $\pi_Y\big|_{X_i \times Y}$ be the restriction of π_Y to $X_i \times Y$, then Lemma 3.1.2 assures that $\pi_Y\big|_{X_i \times Y}(\mathcal{C}) \subseteq Y$ is closed. We can write $\pi_Y(\mathcal{C})$ as $\pi_Y(\mathcal{C}) = \bigcup_i \pi_Y\big|_{X_i \times Y}(\mathcal{C})$. Since $\pi_Y\big|_{X_i \times Y}(\mathcal{C})$ is closed for all $i \in \mathbb{N}$, we conclude that $\pi_Y(\mathcal{C})$ is an F_σ subset of Y. Note that if we have $X_i \times Y \subseteq \mathcal{C}$ for some $i \in \mathbb{N}$, then $\pi_Y\big|_{X_i \times Y}(\mathcal{C}) = Y$ which is automatically closed in Y. □

With a similar argument as we had right after Lemma 3.1.2 we can show that for the projection map $\pi_Y : X \times Y \to Y$ where X is a countable union of compact sets, the image of every F_σ subset is an F_σ subset. Assume that $\mathcal{F} \subset X \times Y$ is a given F_σ subset of $X \times Y$, then there exist closed subsets $\mathcal{C}_i \subset X \times Y$ such that $\mathcal{F} = \bigcup_i \mathcal{C}_i$, where $i \in \mathbb{N}$. We have $\pi_Y(\bigcup_i \mathcal{C}_i) = \bigcup_i \pi_Y(\mathcal{C}_i)$, and Lemma 3.1.3 guarantees that $\pi_Y(\mathcal{C}_i) \subset Y$ is F_σ for all $i \in \mathbb{N}$. Since $\pi_Y(\mathcal{F})$ is a countable union of F_σ subsets of Y, it is itself an F_σ subset.

3.1.2 Introductory definitions and lemmas

For this chapter it is useful to clarify the definition that we have in mind of a topological manifold: A second countable Hausdorff topological space that is locally homeomorphic to an Euclidean space. With this definition, every manifold is a countable union of compact subsets. See Chapter 4 of [Sha17].
Assume that $H : T^*M \to \mathbb{R}$ is a given smooth Hamiltonian defined on the cotangent bundle of a smooth manifold M. Consider $\phi^t(x,u)$ as the Hamiltonian flow associated to $H+u$, where $u \in C^\infty(M)$. With a look back to the statement of Theorem 4, without loss of generality suppose that k is given as 0.
We define $\Delta(x,u) : T^*M \times C^\infty(M) \to C^\infty(M)$ as the projection map to the second variable i.e. the mapping $(x,u) \mapsto u$.

Let $\mathcal{Z}$ be the subset of $T^*M \times C^\infty(M)$ defined as follows

$$\mathcal{Z} := \{(x,u) \in T^*M \times C^\infty(M) \mid (H+u)(x) = 0,\ d(H+u)(x) = 0\}. \tag{3.1.1}$$

Note that $\mathcal{Z}$ is a closed subset of $T^*M \times C^\infty(M)$.
Because T^*M is a countable union of compact subsets and $\mathcal{Z} \subset T^*M \times C^\infty(M)$ is closed, from Lemma 3.1.3 we conclude that $\Delta(\mathcal{Z}) \subset C^\infty(M)$ is an F_σ subset. That is to say the set of potentials $u \in C^\infty(M)$ for which the 0-energy level of $H+u$ is not regular is an F_σ subset of $C^\infty(M)$. Assume that $u_0 \in C^\infty(M)$ is given, then Sard's theorem implies that for any open neighborhood $U_{u_0} \subset C^\infty(M)$ of u_0, there exists $a \in \mathbb{R}^+$ such that $u_0 + a \in U_{u_0}$ and 0 is a regular value of $H+u_0+a$. We just proved one of the assertions of Theorem 4 that is the set of potentials $\{u \in C^\infty(M) \mid (H+u)^{-1}(0) \text{ is a regular energy level}\}$ is a G_δ dense subset of $C^\infty(M)$.

Let Υ be a given nowhere dense F_σ subset of $Sp(2d)$.
We say a periodic orbit is non-degenerate of order one if 1 is not an eigenvalue of its associated linearized restricted Poincaré map. Similarly, a periodic orbit is called non-degenerate of order m if its associated linearized restricted Poincaré map does not take $\sqrt[\ell]{1}$ as an eigenvalue for all $\ell \in \{1,2,\dots,m\}$.
We define subsets $\mathcal{P}^6 \subset \mathcal{P}^5 \subset \mathcal{P}^4 \subset \mathcal{P}^3 \subset \mathcal{P}^2 \subset \mathcal{P}^1 \subset]0,\infty[\times T^*M \times C^\infty(M)$ as following

$$\begin{aligned}
\mathcal{P}^1 &:= \big\{(s,x,u) \in]0,\infty[\times T^*M \times C^\infty(M) \mid x \text{ is a } s\text{-periodic point} \\
&\qquad\qquad \text{of Hamiltonian vector field of } H+u \text{ and } (H+u)(x) = 0\big\},\\
\mathcal{P}^2 &:= \big\{(s,x,u) \in \mathcal{P}^1 \mid x \text{ is a periodic point with minimal period } s\big\},\\
\mathcal{P}^3 &:= \big\{(s,x,u) \in \mathcal{P}^2 \mid t = 0 \text{ is a neat time of the orbit } \theta(t) := \phi^t(x,u)\big\},\\
\mathcal{P}^4 &:= \big\{(s,x,u) \in \mathcal{P}^3 \mid x \notin \mathit{\Gamma}_H\big\},\\
\mathcal{P}^5 &:= \big\{(s,x,u) \in \mathcal{P}^4 \mid\ \theta(t) := \phi^t(x,u) \text{ is non-degenerate of order one}\big\},\\
\mathcal{P}^6 &:= \big\{(s,x,u) \in \mathcal{P}^5 \mid \text{the linearized Poincaré map of } \theta(t) := \phi^t(x,u) \text{ does not belong to } \Upsilon\big\}.
\end{aligned}$$

Note that $\mathcal{P}^1$ contains $]0,\infty[\times\mathcal{Z}$ and it is a closed subset of $]0,\infty[\times T^*M \times C^\infty(M)$. At the other hand, $\mathcal{P}^2$ is disjoint from $]0,\infty[\times\mathcal{Z}$.

The product space $]0,\infty[\times T^*M \times C^\infty(M)$ is a metrizable space. In general, due to Tychonoff [Tyc26], every second countable T_3 (regular Hausdorff) space is metrizable, and a finite product of metrizable spaces is metrizable as well. Here, $]0,\infty[\times T^*M$ is a product of two smooth manifolds, so it is metrizable. In addition, $C^\infty(M)$ is a Férechet space (see Definition 1.6 in [GG73]), so it is metrizable. In conclusion, The product space $]0,\infty[\times T^*M \times C^\infty(M)$ is metrizable.

A subset F of a topological space X is *locally closed* if for each $x \in F$ there exists an open neighborhood $V_x \subset X$ such that $V_x \cap F$ is a closed subset of V_x. Equivalently, $F \subset X$ is locally closed if it is the intersection of a closed and of an open subset of X.

Assume that X is a metrizable topological space, then each locally closed subset of X is an F_σ subset. To illustrate this fact, first note that all closed subsets are F_σ. So it is enough to show that a given open subset $U \subset X$ is F_σ. Suppose that metric d induces the topology of X. For $n \in \mathbb{N}$, define $U_{\frac{1}{n}} := \{B(x, \frac{1}{n}) \mid x \in U\}$, where $B(x, r) := \{y \in X \mid d(y, x) < r\}$ is the open ball of ratio r centered at x. Writing $U_{\frac{1}{n}}$ as $U_{\frac{1}{n}} = \{x \in U \mid d(x, X \setminus U) \geq \frac{1}{n}\}$ shows that $U_{\frac{1}{n}}$ is closed for each $n \in \mathbb{N}$. Since we have $U = \bigcup_{n\in\mathbb{N}} U_{\frac{1}{n}}$, we conclude that U is an F_σ subset of X.

Suppose that X is a topological space and we have $A \subset B \subset X$ in such a way that B is locally closed in X, and A is locally closed in B. We aim to show that A is locally closed in X. Since $B \subset X$ is locally closed, we can write $B = O_1 \cap F_1$ where $F_1 \subset X$ is closed and $O_1 \subset X$ is open. Similarly, there exists an open subset $O_2 \subset X$ and a closed subset $F_2 \subset X$ such that

$$A = (B \cap O_2) \cap (B \cap F_2). \tag{3.1.2}$$

Replacing $B = O_1 \cap F_1$ in equation (3.1.2) above gives $A = (O_1 \cap O_2) \cap (F_1 \cap F_2)$, so A is locally closed in X.

Lemma 3.1.4. *The sets $\mathcal{P}^i \setminus \mathcal{P}^{i+1}$, for $1 \leq i \leq 4$, are F_σ subsets of $]0, \infty[\times T^*M \times C^\infty(M)$.*

Proof. We aim to prove that $\mathcal{P}^1 \setminus \mathcal{P}^2$ is a closed subset of $\mathcal{P}^1$. Then the closed inclusions $\mathcal{P}^1 \setminus \mathcal{P}^2 \subset \mathcal{P}^1 \subset]0, \infty[\times T^*M \times C^\infty(M)$ imply that $\mathcal{P}^1 \setminus \mathcal{P}^2$ is a closed —thus an F_σ— subset of $]0, \infty[\times T^*M \times C^\infty(M)$.
Consider a sequence $(s_k, x_k, u_k) \in \mathcal{P}^1 \setminus \mathcal{P}^2$ converging to $(s, x, u) \in \mathcal{P}^1$. We will prove that $(s, x, u) \notin \mathcal{P}^2$. Denote by S_k the minimal period of x_k. There exists a sequence of integers $i_k \geq 2$ such that $s_k = i_k S_k$. Moreover, there exists an extraction $\phi : \mathbb{N} \to \mathbb{N}$ such that $s_{\phi(k)} = i_{\phi(k)} S_{\phi(k)}$ where $i_{\phi(k)}$ is either a constant sequence or it tends to $+\infty$. In the first mentioned case, if $c \geq 2$ is the constant value of the subsequence $i_{\phi(k)}$, then we have $S_{\phi(k)} \to \frac{s}{c}$. So the point x is $\frac{s}{c}$-periodic which implies that s is not the minimal period of x. Therefore, by definition of $\mathcal{P}^2$, we conclude that $(s, x, u) \notin \mathcal{P}^2$. In the second case, we have $(x, u) \in \mathcal{Z}$ where the subset $\mathcal{Z} \subset T^*M \times C^\infty(M)$ is defined in (3.1.1). Recall that $\mathcal{P}^2$ is disjoint from $]0, \infty[\times \mathcal{Z}$. Hence, in both cases (s, x, u) does not belong to $\mathcal{P}^2$.

We will prove that $\mathcal{P}^2 \setminus \mathcal{P}^3$ is closed in $\mathcal{P}^2$. Assume that $(s_k, x_k, u_k) \in \mathcal{P}^2 \setminus \mathcal{P}^3$ converges to $(s, x, u) \in \mathcal{P}^2$. Denote by $Q_k(t)$ the projected $(H + u_k)$-orbit of x_k and let $Q(t)$ be the limit of $Q_k(t)$. Since 0 is not a neat time for $Q_k(t)$, we can assume by taking a subsequence that either $\dot{Q}_k(0) = 0$ for each k or there exists times $\tau_k \in] - s_k/2, 0[\cup]0, s_k/2]$ such that $Q_k(0) = Q_k(\tau_k)$. The first case immediately gives $\dot{Q}(0) = 0$. In the second situation, if the sequence τ_k has an accumulation point $\tau \neq 0$ then we have $Q(\tau) = Q(0)$. Recalling that s is the minimal period of x and $\tau < s$, the equation $Q(\tau) = Q(0)$ implies that 0 is not a neat time for $Q(t)$. Otherwise, if $\tau_k \to 0$, the equation $Q_k(\tau_k) = Q_k(0)$ gives $\dot{Q}(0) = 0$, so once again 0 is not a neat time for $Q(t)$.

We have proved that $\mathcal{P}^2 \setminus \mathcal{P}^3$ is a closed subset of $\mathcal{P}^2$, so $\mathcal{P}^3$ is open in $\mathcal{P}^2$. Therefore, there exists an open subset $\mathcal{O}_1$ of $]0, \infty[\times T^*M \times C^\infty(M)$ such that $\mathcal{P}^3 = \mathcal{P}^2 \cap \mathcal{O}_1$. Earlier in this proof, we showed that $\mathcal{P}^1 \setminus \mathcal{P}^2$ is closed in $\mathcal{P}^1$ which implies that $\mathcal{P}^2$ is open in $\mathcal{P}^1$, so there exists an open subset $\mathcal{O}_2 \subset]0, \infty[\times T^*M \times C^\infty(M)$ such that $\mathcal{P}^2 = \mathcal{P}^1 \cap \mathcal{O}_2$. Hence, after recalling that $\mathcal{P}^1$ is a closed subset of $]0, \infty[\times T^*M \times C^\infty(M)$ we conclude that $\mathcal{P}^2 = \mathcal{P}^1 \cap \mathcal{O}_2$ and $\mathcal{P}^3 = \mathcal{P}^1 \cap \mathcal{O}_1 \cap \mathcal{O}_2$ are locally closed subsets of $]0, \infty[\times T^*M \times C^\infty(M)$. Since $\mathcal{P}^2 \setminus \mathcal{P}^3$ is closed —thus locally closed— in $\mathcal{P}^2$, and $\mathcal{P}^2 \subset]0, \infty[\times T^*M \times C^\infty(M)$ is locally closed, we conclude that $\mathcal{P}^2 \setminus \mathcal{P}^3$ is a locally closed subset of $]0, \infty[\times T^*M \times C^\infty(M)$. Because $]0, \infty[\times T^*M \times C^\infty(M)$ is metrizable and $\mathcal{P}^2 \setminus \mathcal{P}^3$ is locally closed, $\mathcal{P}^2 \setminus \mathcal{P}^3$ is an F_σ subset of $]0, \infty[\times T^*M \times C^\infty(M)$.

The definitions of $\mathcal{P}^4$ and $\mathcal{P}^5$ are immediately implying that $\mathcal{P}^4$ is open in $\mathcal{P}^3$, and $\mathcal{P}^5$ is open in $\mathcal{P}^4$. Because the inclusions $\mathcal{P}^5 \subset \mathcal{P}^4 \subset \mathcal{P}^3 \subset \mathcal{P}^2 \subset \mathcal{P}^1$ are all open and $\mathcal{P}^1$ is a closed subset of $]0,\infty[\times T^*M \times C^\infty(M)$, a similar reasoning as the above paragraph implies that $\mathcal{P}^5 \setminus \mathcal{P}^4$ and $\mathcal{P}^3 \setminus \mathcal{P}^4$ are both F_σ subsets of $]0,\infty[\times T^*M \times C^\infty(M)$. □

3.2 Proof of the bumpy metric theorem

3.2.1 Proof of Theorem 4

Let us define $\Pi :]0,\infty[\times T^*M \times C^\infty(M) \to C^\infty(M)$ as the projection to the third variable i.e the mapping $(s,x,u) \mapsto u$. Because $]0,\infty[\times T^*M$ is a countable union of compact subsets, Lemma 3.1.3 guarantees that the image of an F_σ subset under the map Π is an F_σ subset of $C^\infty(M)$.
To prove Theorem 4, we will show that $\Pi(\mathcal{P}^4 \setminus \mathcal{P}^6)$ is a nowhere dense F_σ subset of $C^\infty(M)$. That is equivalent to prove that $\Pi(\mathcal{P}^5 \setminus \mathcal{P}^6)$ and $\Pi(\mathcal{P}^4 \setminus \mathcal{P}^5)$ are both nowhere dense F_σ subsets of $C^\infty(M)$. We will accomplish these tasks in Propositions 3.2.1 and 3.2.2 below.
Separability of the product space $]0,\infty[\times T^*M \times C^\infty(M)$ allows us to conclude Proposition 3.2.1 and 3.2.2 after proving them in a neighborhood of a given point $(s_0,x_0,u_0) \in]0,\infty[\times T^*M \times C^\infty(M)$.
Theorem 3 has a central role in the proof of Proposition 3.2.1. Besides, we apply the normal form given in Theorem 1.4.1 in the proof of Proposition 3.2.2.

Proposition 3.2.1. *The set* $\Pi(\mathcal{P}^5 \setminus \mathcal{P}^6)$ *is a nowhere dense* F_σ *subset of* $C^\infty(M)$.

Proof. For a given $(s_0,x_0,u_0) \in]0,\infty[\times T^*M \times C^\infty(M)$, assume that $\mathcal{Y} \subset]0,\infty[\times T^*M \times C^\infty(M)$ is an open neighborhood of (s_0,x_0,u_0). Define $\mathcal{P}^5_{loc} := \mathcal{P}^5 \cap \mathcal{Y}$, and $\mathcal{P}^6_{loc} := \mathcal{P}^6 \cap \mathcal{P}^5_{loc}$. We denote by $(\mathcal{P}^6_{loc})^c$ the complement of $\mathcal{P}^6_{loc} \subset \mathcal{P}^5_{loc}$.
Define $F(s,x,u) : \mathcal{P}^5_{loc} \to \Omega \subset Sp(2d)$ as the map which associates to each periodic point its restricted linearized Poincaré map. Where Ω is the image of $\mathcal{P}^5_{loc}$ under F. By definition of $\mathcal{P}^6$, we have $(\mathcal{P}^6_{loc})^c = F^{-1}(\Upsilon \cap \Omega)$. Because F is continuous, and by Theorem 3 it is weakly open, we conclude that $(\mathcal{P}^6_{loc})^c$ is a nowhere dense F_σ subset of $\mathcal{P}^5_{loc}$.
First order non-degeneracy of periodic orbits has a continuous dependence on the parameter u. Hence, provided by reducing $\mathcal{Y}$ if necessary, the restriction of Π to $\mathcal{P}^5_{loc}$ is a homeomorphism onto its image $\Pi(\mathcal{P}^5_{loc})$ which is an open subset of $\Pi(\mathcal{Y})$. The set $\Pi(\mathcal{P}^5_{loc} \setminus \mathcal{P}^6_{loc})$ can be seen as the image of $(\mathcal{P}^6_{loc})^c$ under the homeomorphism $\Pi\big|_{\mathcal{P}^5_{loc}}$. Therefore, since $(\mathcal{P}^6_{loc})^c$ is a nowhere dense F_σ subset of $\mathcal{P}^5_{loc}$, so is $\Pi(\mathcal{P}^5_{loc} \setminus \mathcal{P}^6_{loc}) \subset \Pi(\mathcal{Y})$.
Because $]0,\infty[\times T^*M \times C^\infty(M)$ is a separable space, we deduce that $\Pi(\mathcal{P}^5 \setminus \mathcal{P}^6)$ is a nowhere dense F_σ subset of $C^\infty(M)$. □

Proposition 3.2.2. *The set* $\Pi(\mathcal{P}^4 \setminus \mathcal{P}^5)$ *is a nowhere dense* F_σ *subset of* $C^\infty(M)$.

Proof. We have already proved that $\Pi(\mathcal{P}^4 \setminus \mathcal{P}^5) \subset C^\infty(M)$ is F_σ in Lemma 3.1.4, so it remains to show that $\Pi(\mathcal{P}^4 \setminus \mathcal{P}^5)$ is a nowhere dense subset of $C^\infty(M)$. Let $(s_0,x_0,u_0) \in \mathcal{P}^4$ be given where $x_0 = (q_0,p_0)$. Without loss of generality assume that $u_0 = 0$, and $x_0 = 0$ is the origin of the local coordinates given in Theorem 1.4.1. We consider the section $\{q_1 = 0\}$ which is transverse to $\theta(t) := \phi^t(x_0,u_0)$ at $\theta(0)$. Due to properties of the local coordinates, $\theta(t)$ has zero energy; Furthermore, since $dH(0) = (e_1, 0)$, the tangent space to $\{H = 0\}$ at the point $x_0 = 0$ is $\{p_1 = 0\}$.
If u belongs to a sufficiently small neighborhood $C^\infty_{loc}(M) \subset C^\infty(M)$ of $u = 0$, then $\hat{x} = (\hat{q},\hat{p})$ are symplectic local coordinates of the section

$$\Lambda(u) := \big\{(q,p) \in \mathbb{R}^{2d+2} \mid q_1 = 0,\ (H+u)(q,p) = 0\big\}.$$

Let $\mathbb{R}^{2d}_{loc} \subset \big\{\hat{x} \mid x(\hat{x}, u) \in \Lambda(u)\big\}$ be an open neighborhood around $\hat{x} = 0$, where $x(\hat{x}, u) \in \Lambda(u)$ denotes for the point that has coordinates $\hat{x}$.
Consider

$$\tau(\hat{x}, u) : \mathbb{R}^{2d}_{loc} \times C^\infty_{loc}(M) \to \mathbb{R}$$

as the first return time and

$$\psi(\hat{x}, u) : \mathbb{R}^{2d}_{loc} \times C^\infty_{loc}(M) \to \mathbb{R}^{2d}_{loc}$$

as the first return map. In fact, $\psi(\hat{x}, u)$ is the $\hat{x}$ coordinates of the point $\phi\big(\tau(\hat{x}, u), x(\hat{x}, u), u\big)$ where $x(\hat{x}, u) \in \Lambda(u)$.
Let $\mathcal{S}$ be the set of solutions of the equation

$$\psi(\hat{x}, u) = \hat{x}. \tag{3.2.1}$$

Moreover, define $\mathcal{S}^5$ as the set of non-degenerate solutions of (3.2.1); More precisely,

$$\mathcal{S}^5 := \big\{(\hat{x}, u) \in \mathcal{S} \mid 1 \text{ is not an eigenvalue of } \partial_{\hat{x}}\psi(\hat{x}, u)\big\}.$$

Note that $(\hat{x}, u) \in \mathcal{S}$ if and only if $(\tau(\hat{x}, u), x(\hat{x}, u), u) \in \mathcal{P}^1$. By Lemma 3.1.4, the inclusion $\mathcal{P}^4 \subset \mathcal{P}^1$ is open, so up to reducing $\mathbb{R}^{2d}_{loc}$ and $C^\infty_{loc}(M)$ we conclude that $(\hat{x}, u) \in \mathcal{S}$ if and only if $(\tau(\hat{x}, u), x(\hat{x}, u), u) \in \mathcal{P}^4$.
We define

$$\mathcal{P}^4_{loc} := \mathcal{P}^4 \cap \big\{\big(\tau(\hat{x}, u), \phi^t(x(\hat{x}, u), u), u\big) \mid t \in \mathbb{R},\ (\hat{x}, u) \in \mathcal{S}\big\}$$

which is an open neighborhood of $(s_0, 0, 0) \in \mathcal{P}^4$. By definition of $\mathcal{P}^4_{loc}$, we have

$$\Pi(\mathcal{P}^4_{loc}) = \Delta(\mathcal{S}),$$

where Π and Δ are projections to the third variable and to the second variable respectively. Since $(\hat{x}, u) \in \mathcal{S}^5$ if and only if $\big(\tau(\hat{x}, u), x(\hat{x}, u), u\big) \in \mathcal{P}^5$, we conclude that

$$\Pi(\mathcal{P}^4_{loc} \setminus \mathcal{P}^5) = \Delta(\mathcal{S} \setminus \mathcal{S}^5).$$

So we reduced the claim of the proposition to:

$$\Delta(\mathcal{S} \setminus \mathcal{S}^5) \text{ is a nowhere dense subset of } C^\infty(M). \tag{3.2.2}$$

Our strategy to prove the claim (3.2.2) above is to show that $\mathcal{S}$ is a submanifold of $\mathbb{R}^{2d}_{loc} \times C^\infty_{loc}(M)$, and $\mathcal{S}^5$ is the set of regular points of the map $\Delta\big|_{\mathcal{S}}$ i.e. the restriction of the projection map Δ to $\mathcal{S}$. Applying Sard's theorem then completes the proof. However, first we wish to restrict our setup to a finite dimensional space of $C^\infty(M)$ which allows us to use the notion of Fréchet differential instead of Gateau differential with respect to u.

Lemma 3.2.3. *For a given open neighborhood $U \subset M$ of q_0, there exists a finite dimensional subspace $E \subset C^\infty(M)$ formed by potentials supported inside U and null on the orbit θ, such that $\partial_u\psi(0, 0)$ sends E onto $\mathbb{R}^{2d}$.*

Assuming the above lemma, we finish the proof of Proposition 3.2.1 considering ∂_u as the notion of Fréchet derivative. By Lemma 3.2.3, $\partial_u\psi(0, 0)E = \mathbb{R}^{2d}$. Therefore, up to reducing the neighborhoods $\mathbb{R}^{2d}_{loc}$ and $C^\infty_{loc}(M)$ if necessary, we have

$$\partial_u\psi(\hat{x}, u)E = \mathbb{R}^{2d}, \quad \text{for all } (\hat{x}, u) \in \mathbb{R}^{2d}_{loc} \times C^\infty_{loc}(M).$$

Let $E_{loc} \subset E$ be a neighborhood of 0. For a given $v \in C^\infty_{loc}(M)$, define $\Psi_v : \mathbb{R}^{2d}_{loc} \times E_{loc} \to \mathbb{R}^{2d}$ as

$$\Psi_v(\hat{x}, u) := \psi(\hat{x}, u + v) - \hat{x}.$$

If E_{loc} is a sufficiently small neighborhood of 0, then Ψ_v is a submersion. In consequence, the set $N := \Psi_v^{-1}(0)$ is a submanifold, and $\{(\hat{x}, u) \in N \mid \partial_{\hat{x}}\Psi_v(\hat{x}, u) \text{ is not invertible}\}$ is the set of singular points of the map $\Delta\big|_N$; See Proposition 2.2 of [BM15].
If $v + u$ belongs to $\Delta(\mathcal{S} \setminus \mathcal{S}^5)$, then u is a critical value of $\Delta\big|_N$: Suppose that $v + u \in \Delta(\mathcal{S} \setminus \mathcal{S}^5)$, then there exists $\hat{x} \in \mathbb{R}^{2d}_{loc}$ such that $(\hat{x}, v + u) \in (\mathcal{S} \setminus \mathcal{S}^5)$, so $(\hat{x}, u) \in N$ is a critical point of $\Delta\big|_N$ which implies that u is a critical value of $\Delta\big|_N$. By Sard's theorem, there exist regular values of $\Delta\big|_N$ arbitrarily close to 0, so v does not belong to the interior of $\Delta(\mathcal{S} \setminus \mathcal{S}^5)$. Since this holds for all $v \in C^\infty_{loc}(M)$, we conclude that $\Delta(\mathcal{S} \setminus \mathcal{S}^5)$ is nowhere dense. □

Assume that $\partial_u\phi^t(x, u)(h)$, where $h \in C^\infty(M)$, is the Gateau differential of $\phi^t(x, u)$ with respect to u at the point $(t, x, u) \in]0, \infty[\times T^*M \times C^\infty(M)$. We have

$$\partial_u\phi^t(x, u)(h) = \partial_x\phi^t(x, u) \int_0^t [\partial_x\phi^s(x, u)]^{-1} \begin{bmatrix} 0 \\ -dh\big(\pi \circ \phi^s(x, 0)\big) \end{bmatrix} ds. \tag{3.2.3}$$

In order to obtain (3.2.3), note that $\partial_u\phi^t(x, u)(h)$ satisfies the differential equation

$$\partial_t\partial_u\phi^t(x, u)(h) = \mathbb{J}\partial^2_{x^2}(H + u)\big(\phi^t(x, u)\big)\partial_u\phi^t(x, u)(h) + \begin{bmatrix} 0 \\ -dh\big(\pi \circ \phi^t(x, 0)\big) \end{bmatrix}. \tag{3.2.4}$$

We are able to verify equation (3.2.4) above via the following computations

$$\begin{aligned} \partial_t\partial_u\phi^t(x, u) = \partial_u\partial_t\phi^t(x, u) &= \partial_u\big[\mathbb{J}d(H + u)\big(\phi^t(x, u)\big)\big] \\ &= \partial_x\big[\mathbb{J}d(H + u)\big(\phi^t(x, u)\big)\big]\partial_u\phi^t(x, u) \\ &+ \partial_u[\mathbb{J}d(H + u)\big(\phi^t(x, u)\big)] \\ &= \mathbb{J}\partial^2_{x^2}(H + u)\big(\phi^t(x, u)\big)\partial_u\phi^t(x, u) + \begin{bmatrix} 0 \\ -dh\big(\pi \circ \phi^t(x, 0)\big) \end{bmatrix}. \end{aligned}$$

Furthermore, we have

$$\partial_t\partial_x\phi^t(x, u) = \mathbb{J}\partial^2_{x^2}(H + u)\big(\phi^t(x, u)\big)\partial_x\phi^t(x, u), \tag{3.2.5}$$

which is similar to equation (2.2.3) in the previous chapter. Note that the differential equation (3.2.5) is the non-homogeneous equation associated to (3.2.4), so equation (3.2.3) would be the result of the well-known relation between solutions of a linear non-homogeneous ordinary differential equation and its corresponding homogeneous equation. Once again we refer to Chapter IV of Hartman [Har82], Corollary 2.1.

Proof of Lemma 3.2.3. We work in the local coordinates given by Theorem 1.4.1 around $x = 0$, so $\mathbb{J}\partial^2_{x^2}H(te_1, 0)$ has the block form alike (2.2.1). Therefore, equation (3.2.3) is uncoupled with respect to $x_1 = (q_1, p_1)$ and $\hat{x} = (\hat{q}, \hat{p})$ coordinates.
Let $\psi^t(\hat{x}, u) : \{q_1 = 0,\ H + u = 0\} \to \{q_1 = t,\ H + u = 0\}$ be the one-parameter family of restricted transition maps between $\{q_1 = 0\}$ and $\{q_1 = t\}$. Note the difference between the notations $\psi(\hat{x}, u)$ and $\psi^t(\hat{x}, u)$ which are referring to the return map and the one-parameter family of transition maps respectively.

$\psi^t(\hat{x},u)$ is the $\hat{x}$-coordinates of $\phi^t\big(x(\hat{x},u),u\big)$ where $x(\hat{x},u)\in\Lambda(u)$ denotes for the point with coordinates $\hat{x}$. Therefore, for a fixed σ near 0

$$\partial_u\psi^\sigma(0,0)(h)=\partial_{\hat{x}}\phi^\sigma(0,0)\int_0^\sigma[\partial_{\hat{x}}\phi^s(0,0)]^{-1}\begin{bmatrix}0_d\\-\partial_{\hat{q}}h(se_1)\end{bmatrix}ds.$$

Take $\eta_\epsilon(t)$ supported in $]0,\sigma[$ as an smooth approximation of Dirac delta distribution $\delta(t)$. More precisely, $\lim_{\epsilon\to0^+}\eta_\epsilon(t)=\delta(t)$ where the limit is in the sense of distribution. For $1\le i,j\le d$, let $h_{ij}\in C^\infty(M)$ be given such that

$$-\partial_{\hat{q}}h_{ij}(te_1)=\eta_\epsilon(t)e_j+\eta'_\epsilon(t)D^{-1}e_i,$$

where $\eta'_\epsilon(t)$ is the derivative with respect to t of $\eta_\epsilon(t)$, and $D=\partial^2_{\hat{p}^2}H(te_1,0)$ is a constant diagonal matrix with only $+1$ or -1 entries; Look at the block form expression of $\mathbb{J}\partial^2_{x^2}H(te_1,0)$ in (2.2.1) again. For this proof we only need to use the fact that D is invertible.
Provided $\epsilon>0$ be sufficiently near zero, we show that $\partial_u\psi^\sigma(0,0)(h_{ij})$ is a basis for $\mathbb{R}^{2d}$. After setting $V(t):=\partial_{\hat{x}}\phi^t(0,0)$, we have

$$\begin{aligned}\partial_u\psi^\sigma(0,0)(h_{ij})&=V(\sigma)\int_0^\sigma V^{-1}(s)\begin{bmatrix}0\\\eta_\epsilon(s)e_j+\eta'_\epsilon(s)D^{-1}e_i\end{bmatrix}ds\\&=V(\sigma)\int_0^\sigma V^{-1}(s)\begin{bmatrix}0\\\eta_\epsilon(s)e_j\end{bmatrix}ds+V(\sigma)\int_0^\sigma V^{-1}(s)\begin{bmatrix}0\\\eta'_\epsilon(s)D^{-1}e_i\end{bmatrix}ds.\end{aligned}\tag{3.2.6}$$

It is clear that we have

$$V(\sigma)\int_0^\sigma V^{-1}(s)\eta_\epsilon(s)\begin{bmatrix}0\\e_j\end{bmatrix}ds\approx V(\sigma)V^{-1}(0)\begin{bmatrix}0\\e_j\end{bmatrix}.\tag{3.2.7}$$

Now we compute the other term $V(\sigma)\int_0^\sigma V^{-1}(s)\begin{bmatrix}0\\\eta'_\epsilon(s)D^{-1}e_j\end{bmatrix}ds$, in the right hand side of the equation (3.2.6):

$$\begin{aligned}V(\sigma)\int_0^\sigma V^{-1}(s)\begin{bmatrix}0\\\eta'_\epsilon(s)D^{-1}e_i\end{bmatrix}ds&=-V(\sigma)\int_0^\sigma -V^{-1}(s)\mathbb{J}\partial^2_{\hat{x}^2}H(se_1,0)\begin{bmatrix}0\\\eta_\epsilon(s)D^{-1}e_i\end{bmatrix}ds\\&=V(\sigma)\int_0^\sigma V^{-1}(s)\begin{bmatrix}0&D\\-K(s)&0\end{bmatrix}\begin{bmatrix}0\\\eta_\epsilon(s)D^{-1}e_i\end{bmatrix}ds\\&=V(\sigma)\int_0^\sigma V^{-1}(s)\eta_\epsilon(s)\begin{bmatrix}e_i\\0\end{bmatrix}ds\\&\approx V(\sigma)V^{-1}(0)\begin{bmatrix}e_i\\0\end{bmatrix}.\end{aligned}\tag{3.2.8}$$

Equations (3.2.6), (3.2.7) and (3.2.8) imply that $\partial_u\psi^\sigma(0,0)(h_{ij})\approx V(\sigma)V^{-1}(0)\begin{bmatrix}e_i\\e_j\end{bmatrix}$. In conclusion, because $V(\sigma)V^{-1}(0)$ is invertible, $\partial_u\psi^\sigma(0,0)(h_{ij})$ forms a basis of $\mathbb{R}^{2d}$.

Consider the map $G_\sigma(\hat{x})$ as the restricted transition map from $\{q_1=\sigma\}$ to $\{q_1=0\}$ along the periodic orbit. For potentials u that their supports are disjoint from $\{te_1\mid t\in[\sigma,s_0]\}$ we can write the restricted Poincaré map $\psi(\hat{x},u)$ as

$$\psi(\hat{x},u)=G_\sigma\big(\psi^\sigma(\hat{x},u)\big).$$

Then, because $\partial_u \psi^\sigma(0,0)$ is onto we conclude that $\partial_u \psi(0,0) = \partial_{\hat{x}} G_\sigma(0) \circ \partial_u \psi^\sigma(0,0)$ is onto. □

One of the key points of the proof of Lemma 3.2.3 is that the matrix $\partial^2_{\tilde{p}^2} H(se_1, 0)$ viewed in the coordinates given in Theorem 1.4.1 is invertible for all $s \in [0,\sigma]$. That is precisely the expression in coordinates of fiberwise iso-energetically non-degeneracy at 0.

3.2.2 Proof of Theorem 5

We prove Theorem 5 which is the last step that we need to take to achieve the bumpy metric theorem. For a given $n \in \mathbb{N}$ we define

$$\mathcal{R}(n) := \big\{(x,u) \in T^*M \times C^\infty(M) \mid \partial_p H(x) \neq 0,\ \phi([-\frac{1}{n}, \frac{1}{n}] \times \{x\} \times \{u\}) \subset \varGamma_H\big\},$$

where occasionally we have preferred the notation $\phi(t,x,u)$ over $\phi^t(x,u)$ which we have frequently used during this thesis. Proving Theorem 5 is the matter of showing that $\Delta\big(\bigcup_{n\in\mathbb{N}} \mathcal{R}(n)\big)$ is a nowhere dense F_σ subset of $C^\infty(M)$.
By definition, for a given $n \in \mathbb{N}$, $\mathcal{R}(n)$ is a locally closed subset of $T^*M \times C^\infty(M)$. Note that $\partial_p H(x) \neq 0$ is an open condition for the points $(x,u) \in T^*M \times C^\infty(M)$, and for a given $n \in \mathbb{N}$,

$$\big\{(x,u) \mid \phi([-\frac{1}{n}, \frac{1}{n}] \times \{x\} \times \{u\}) \subset \varGamma_H\big\}$$

is a closed subset of $T^*M \times C^\infty(M)$. For a given $n \in \mathbb{N}$, since $\mathcal{R}(n) \subset T^*M \times C^\infty(M)$ is locally closed and $T^*M \times C^\infty(M)$ is metrizable, we conclude that $\mathcal{R}(n)$ is an F_σ subset of $T^*M \times C^\infty(M)$. In consequence, $\Delta\big(\bigcup_{n\in\mathbb{N}} \mathcal{R}(n)\big) = \bigcup_{n\in\mathbb{N}} \Delta\big(\mathcal{R}(n)\big) \subset C^\infty(M)$ is an F_σ subset. Where we used the fact that the image of an F_σ subset under Δ is F_σ, and union of countable F_σ subsets is F_σ.

It remains to show that $\Delta\big(\bigcup_{n\in\mathbb{N}} \mathcal{R}(n)\big)$ is nowhere dense which we conclude after proving that $\Delta\big(\mathcal{R}(n)\big)$ is nowhere dense for a given $n \in \mathbb{N}$. Because $T^*M \times C^\infty(M)$ is separable, assuming that $n \in \mathbb{N}$ is given, it would be enough to show that for a given $(x_0,u_0) \in \mathcal{R}(n)$ neighborhoods $(T^*M)_{loc} \subset T^*M$ of x_0 and $C^\infty_{loc}(M)$ of u_0 exists such that $\Delta\Big(\big((T^*M)_{loc} \times C^\infty_{loc}(M)\big) \cap \mathcal{R}(n)\Big)$ is nowhere dense.

Let $n \in \mathbb{N}$, and $k > 2d+2$ be given. Consider

$$0 < \sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_k < \sigma_{k+1} < \frac{1}{n}.$$

Define $\Phi : T^*M \times C^\infty(M) \to (T^*M)^k$ as

$$\Phi(x,u) := \big(\phi(\sigma_1,x,u), \ldots, \phi(\sigma_k,x,u)\big).$$

The lemma that follows is the key to prove Theorem 5.

Lemma 3.2.4. *Assume that $(x_0,u_0) \in T^*M \times C^\infty(M)$ such that $\partial_p H(x_0) \neq 0$. The times σ_i, where $0 \le i \le k+1$, can be chosen in a way that a finite dimensional subspace $E \subset C^\infty(M)$ exists for which the map $F_{u_0} : T^*M \times E \to (T^*M)^k$ defined as $F_{u_0}(x,u) := \Phi(x, u_0 + u)$ is transverse to $(\varGamma_H)^k$ at the point $(x_0, 0)$.*

With assuming the above lemma, we give a proof of Theorem 5. Suppose a point $(x_0,u_0) \in T^*M \times C^\infty(M)$ such that $\partial_p H(x_0) \neq 0$ is given. By the above lemma, there exists a finite dimensional subspace $E \subset C^\infty(M)$ so that the map $F_{u_0}(x,u)$ is transverse to $(\varGamma_H)^k$ at $(x_0, 0)$. Because transversality is an open property, there exists neighborhoods $(T^*M)_{loc}$ of x_0

and $C^\infty_{loc}(M)$ of u_0 such that the map $F_{u_1}(x,u)$ is transverse to $(\mathit{\Gamma}_H)^k$ at each point $(x_1,0)$ whenever $(x_1,u_1)\in(T^*M)_{loc}\times C^\infty_{loc}(M)$. Therefore, $F^{-1}_{u_1}\big((\mathit{\Gamma}_H)^k\big)$ is a submanifold of $(T^*M)_{loc}\times E$; Moreover, the codimension $F^{-1}_{u_1}\big((\mathit{\Gamma}_H)^k\big)$ in $(T^*M)_{loc}\times E$ is the same as the codimension of $(\mathit{\Gamma}_H)^k\subset(T^*M)^k$. Note that we have used a well known conclusion of the *preimage theorem*, look at page 28 of [Gui+74]. By assumption, $\mathit{\Gamma}_H\subset T^*M$ has positive codimension which implies that the codimension of $(\mathit{\Gamma}_H)^k$ in $(T^*M)^k$ is at least k. So codimension of $F^{-1}_{u_1}\big((\mathit{\Gamma}_H)^k\big)\subset(T^*M)_{loc}\times E$ is at least k. Therefore, because $(T^*M)_{loc}$ has dimension $2d+2$, and we have $k>2d+2$, by Sard's theorem the projection of $F^{-1}_{u_1}\big((\mathit{\Gamma}_H)^k\big)$ to E is nowhere dense. Hence, there exists $\tilde{u}\in E$ arbitrary close to 0 such that $\tilde{u}$ does not belong to $\Delta\big(F^{-1}_{u_1}\big((\mathit{\Gamma}_H)^k\big)\big)$. For such $\tilde{u}$, there is not exists $x\in(T^*M)_{loc}$ such that $\Phi(x,u_1+\tilde{u})\in(\mathit{\Gamma}_H)^k$. Accordingly, for such $\tilde{u}$ there is not exists $x\in(T^*M)_{loc}$ such that $(x,u_1+\tilde{u})\in\mathcal{R}(n)$; Since that holds for each $u_1\in C^\infty_{loc}(M)$ we conclude that $\Delta\big(\big((T^*M)_{loc}\times C^\infty_{loc}(M)\big)\cap\mathcal{R}(n)\big)$ is nowhere dense.

Proof of Lemma 3.2.4. Without loss of generality we assume that $u_0=0$. Because $\partial_pH(x_0)\neq0$, there exists $\delta>0$, and a local coordinates around x_0 such that it maps x_0 to 0_{2d+2} and we have $\phi^t(0,0)=(te_1,0)$ in the local coordinates for all $t\in[-\delta,\delta]$. Note that H is not necessarily fiberwise iso-energetically non-degenerate at x_0, that is why here in this proof —in contrary to the proof of Lemma 3.2.3— we are not able to use the local coordinates given in Theorem 1.4.1. However, with a review of the proofs of assertions (1) and (2) of Theorem 1.4.1, we recall that the condition $\partial_pH(x_0)\neq0$ (which is weaker than $x_0\notin\mathit{\Gamma}_H$) is enough to conclude the assertions (1) and (2) of Theorem 1.4.1.
Because x_0 is a neat point, we can reduce δ if necessary in order to assure that $\pi\circ\theta(t)$ has no self-intersection, where $t\in[0,\delta]$. Afterwards, we choose $k+1$ distinct $\sigma_i\in[0,\delta]$, for $1\leq i\leq k+1$.

Based on equation (3.2.3), we have

$$\partial_u\phi^t(0,0)(h)=\partial_x\phi^t(0,0)\int_0^t[\partial_x\phi^s(0,0)]^{-1}\begin{bmatrix}0\\-dh\big(\pi\circ\phi^s(0,0)\big)\end{bmatrix}ds.$$

Consider $\eta^i_\epsilon(t)$ as the approximation of the Dirac delta distribution at the time σ_i, namely $\delta_i(t)$, in a way that $supp\,\eta^i_\epsilon\subset]\sigma_{i-1},\sigma_i[$. That means as ϵ approaches σ^i from left, the limit of $\eta^i_\epsilon(t)$ in the sense of distribution is equal to $\delta_i(t)$.
Let $h_{ij}\in C^\infty(M)$ be given such that

$$-dh_{ij}(te_1)=\eta^i_\epsilon(t)e_j,$$

where for $1\leq j\leq d+1$, e_j is the standard basis of $\mathbb{R}^{d+1}$. Define $l_j:=(0,e_j)\in\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}$, and $W(t):=\partial_x\phi^t(0,0)$, then we have

$$\partial_u\phi^{\sigma_i}(0,0)(h_{ij})\approx l_j.$$

Furthermore, for all $i'>i$, we have

$$\partial_u\phi^{\sigma_{i'}}(0,0)(h_{ij})\approx W(\sigma_{i'})W^{-1}(\sigma_i)l_j.$$

So we have

$$\partial_u\Phi(0,0)(h_{ij})\approx\big(\overbrace{0_{2d+2},\ldots,0_{2d+2}}^{i-1},l_j,W(\sigma_{i+1})W^{-1}(\sigma_i)l_j,\ldots,W(\sigma_k)W^{-1}(\sigma_i)l_j\big)=:\vartheta_{ij}.$$

Since $W(t)$ is invertible for all $t\in[0,\delta]$, the vectors $\vartheta_{ij}\in(\mathbb{R}^{2d+2})^k$ are making a basis for the subspace $(0_{d+1}\times\mathbb{R}^{d+1})^k$ which is transverse to $(\mathit{\Gamma}_H)^k$. □

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Colophon

Mémoire de thèse intitulé « Théorème des métriques bosselées au sens de Mañé pour les champs de vecteurs Hamiltonien non convexe», écrit par Shahriar Aslani, achevé le juin 2022, composé au moyen du système de préparation de document LaTeX et de la classe yathesis dédiée aux thèses préparées en France.

Théorème des métriques bosselées au sens de Mañé pour les champs de vecteurs Hamiltonien non convexe

Résumé

Une propriété est générique au sensé de Mañé si, donné un Hamiltonien $H : T^*M \to \mathbb{R}$, l'ensemble des fonctions lisses $u : M \to \mathbb{R}$ tel que $H + u$ vérifie la propriété est un sous-ensemble générique de $C^\infty(M)$. Notre objectif est de savoir dans quelle mesure la non dégénérescence de toutes les orbites périodiques dans un niveau d'énergie donné d'un Hamiltonien lisse non convexe est une propriété générique au sensé de Mañé. Où la non-dégénérescence signifie que dérivée de l'application de Poincaré ne prend pas les racines de l'unité comme une valeurs propre.
Pour atteindre cet objectif, nous obtiendrons un théorème de perturbation pour les aplication de Poincaré similaire au théorème de Rifford et Ruggiero dans le cadre convexe, et une forme normale de type Fermi sur les orbites d'un champ de vecteurs Hamiltonien non convexe. Ce sont deux outils applicables à l'étude de la dynamique des champs de vecteurs Hamiltoniens non convexes. D'autre part, nous montrerons que dans les cas convexes et non convexes, nous avons certainement besoin d'un mécanisme différent pour prouver le théorème des métrique bosselées pour les orbites symétriques. Une orbite symétrique est une orbite dont la projection sur les variétés de base comprend soit des points d'auto-intersection, soit des points à vitesse nulle. Ce fait a été négligé dans les études précédentes.
Une étude détaillée des formes normales locales sur les segments d'orbite d'un champ de vecteurs Hamiltonien est donnée. Cela inclut une forme normale pour les Hamiltoniens convexes, une forme normale pour les Hamiltoniens positivement homogènes qui implique la forme normale de Li-Nienberg pour les métriques de Finsler, et comme nous l'avons mentionné une forme normale pour les Hamiltoniens non convexes. De cette façon, nous éliminons la confusion qui existe dans la littérature entre la forme normale de Li-Nirenberg et une forme normale souhaitée similaire pour les champs de vecteurs Hamiltoniens convexes.

Mots clés : Hamiltonien non convexe, théorème des métriques bosselées, généricité au sens de Mañé

Bumpy metric theorem in the sense of Mañé for non-convex Hamiltonian vector fields

Abstract

A property (p) of smooth Hamiltonian vector fields is called Mañé-generic whenever the set of smooth potentials u such that $H + u$ satisfies the property (p) is a generic subset.
Given a non-convex smooth Hamiltonian $H : T^*M \to \mathbb{R}$ which is defined on the cotangent bundle of a smooth manifold M, our goal in this thesis is to know that to what extent non-degeneracy of all periodic orbits in a given energy level of H is a Mañé generic property. Where by a periodic non-degenerate orbit we mean a periodic orbit that its associated linearized Poincaré map does not take roots of unity as an eigenvalue.
To that end, we will achieve a perturbation theorem for linearized Poincaré maps similar to Rifford and Ruggiero's theorem in the convex setting, and a Fermi type normal form on orbits of a non-convex Hamiltonian vector field. These are two applicable tools in the study of non-convex Hamiltonian vector fields. At the other hand, we will show that in both convex and non-convex cases we certainly need a different machinery to prove the bumpy metric theorem for symmetric orbits. A symmetric orbit is an orbit that its projection on the base manifolds includes either self-intersection points or points with zero velocity. This fact was overlooked in previous studies.
A detailed study of local normal forms on orbit segments of a Hamiltonian vector field is given. That includes a normal form for convex Hamiltonians, a normal form for positively homogeneous Hamiltonians that implies Li-Nienberg normal form for Finsler metrics, and as we mentioned a normal form for non-convex Hamiltonians. In this way, we remove the confusion that exists in the literature between Li-Nirenberg normal form and a similar desired normal form for convex Hamiltonian vector fields.

Keywords: non-convex Hamiltonians, bumpy metric theorem, Mañé-generic properties

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ÉCOLE NORMALE SUPÉRIEURE

RÉSUMÉ

Une propriété est générique au sensé de Mañé si, donné un Hamiltonien $H : T^*M \to \mathbb{R}$, l'ensemble des fonctions lisses $u : M \to \mathbb{R}$ tel que $H + u$ vérifie la propriété est un sous-ensemble générique de $C^\infty(M)$.
Notre objectif est de savoir dans quelle mesure la non dégénérescence de toutes les orbites périodiques dans un niveau d'énergie donné d'un Hamiltonien lisse non convexe est une propriété générique au sensé de Mañé. Où la non-dégénérescence signifie que dérivée de l'application de Poincaré ne prend pas les racines de l'unité comme une valeurs propre.
Pour atteindre cet objectif, nous obtiendrons un théorème de perturbation pour les aplication de Poincaré similaire au théorème de Rifford et Ruggiero dans le cadre convexe, et une forme normale de type Fermi sur les orbites d'un champ de vecteurs Hamiltonien non convexe. Ce sont deux outils applicables à l'étude de la dynamique des champs de vecteurs Hamiltoniens non convexes. D'autre part, nous montrerons que dans les cas convexes et non convexes, nous avons certainement besoin d'un mécanisme différent pour prouver le théorème des métrique bosselées pour les orbites symétriques. Une orbite symétrique est une orbite dont la projection sur les variétés de base comprend soit des points d'auto-intersection, soit des points à vitesse nulle. Ce fait a été négligé dans les études précédentes.
Une étude détaillée des formes normales locales sur les segments d'orbite d'un champ de vecteurs Hamiltonien est donnée. Cela inclut une forme normale pour les Hamiltoniens convexes, une forme normale pour les Hamiltoniens positivement homogènes qui implique la forme normale de Li-Nienberg pour les métriques de Finsler, et comme nous l'avons mentionné une forme normale pour les Hamiltoniens non convexes. De cette façon, nous éliminons la confusion qui existe dans la littérature entre la forme normale de Li-Nirenberg et une forme normale souhaitée similaire pour les champs de vecteurs Hamiltoniens convexes.

MOTS CLÉS

Hamiltonien non convexe, Théorème des métriques bosselées, Généricité au sens de Mañé

ABSTRACT

A property (p) of smooth Hamiltonian vector fields is called Mañé-generic whenever the set of smooth potentials u such that $H + u$ satisfies the property (p) is a generic subset.
Given a non-convex smooth Hamiltonian $H : T^*M \to \mathbb{R}$ which is defined on the cotangent bundle of a smooth manifold M, our goal in this thesis is to know that to what extent non-degeneracy of all periodic orbits in a given energy level of H is a Mañé generic property. Where by a periodic non-degenerate orbit we mean a periodic orbit that its associated linearized Poincaré map does not take roots of unity as an eigenvalue.
To that end, we will achieve a perturbation theorem for linearized Poincaré maps similar to Rifford and Ruggiero's theorem in the convex setting, and a Fermi-like normal form on orbits of a non-convex Hamiltonian vector field. These are two applicable tools in the study of non-convex Hamiltonian vector fields. At the other hand, we will show that in both convex and non-convex cases we certainly need a different machinery to prove the bumpy metric theorem for symmetric orbits. A symmetric orbit is an orbit that its projection on the base manifolds includes either self-intersection points or points with zero velocity. This fact was overlooked in previous studies.
A detailed study of local normal forms on orbit segments of a Hamiltonian vector field is given. That includes a normal form for convex Hamiltonians, a normal form for positively homogeneous Hamiltonians that implies Li-Nienberg normal form for Finsler metrics, and a normal form for non-convex Hamiltonians. In this way, we remove the confusion that exists in the literature between Li-Nirenberg normal form and a similar desired normal form for convex Hamiltonian vector fields.

KEYWORDS

Non-convex Hamiltonians, Bumpy metric theorem, Mañé generic properties